



Frontiers in Entropy

ISSN: 2989-5413 (Online)

Publisher: Tomorrow's Research Publications (TRP)

An International Peer-Reviewed Open Access Journal

Journal homepage: <https://trpubonline.org/index.php/Entropy>

RESEARCH ARTICLE

Kolmogorov-Sinai Entropy in Nonlinear Dynamical Systems

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ABSTRACT

Kolmogorov-Sinai (KS) entropy quantifies the rate at which a chaotic dynamical system generates information, and Pesin's theorem relates it to the sum of positive Lyapunov exponents, but practical estimation of KS entropy from finite, noisy trajectory data remains substantially less reliable than computing Lyapunov exponents directly, particularly in spatially extended systems. We computed KS entropy via three independent routes, direct Lyapunov-exponent summation under Pesin's theorem, symbolic-dynamics block entropy, and a Grassberger-Procaccia correlation-entropy estimator, for a one-dimensional coupled map lattice of logistic maps with 8 to 128 lattice sites and coupling strengths from 0.0 (uncoupled) to 0.5, across 65 parameter combinations, each analyzed from time series of 2×10^5 iterations after transient removal. In the uncoupled limit, all three estimators agreed with the Pesin-theorem value to within 3% at the fully chaotic control parameter ($r = 4.0$), but Grassberger-Procaccia correlation-entropy estimates diverged sharply as lattice size increased, underestimating true KS entropy by 41% at 128 sites and moderate coupling (0.2), a consequence of the curse of dimensionality afflicting correlation-integral estimation in high-dimensional phase space. Symbolic-dynamics block entropy tracked the Pesin-theorem value more robustly, remaining within 12% even at 128 sites, but required block lengths that grew computationally prohibitive ($>10^7$ distinct symbol blocks) beyond 64 sites, imposing a practical system-size ceiling on this method distinct from its accuracy limitations. Adding observational noise at signal-to-noise ratios from 40 dB down to 10 dB degraded the correlation-entropy estimator fastest, with error exceeding 50% below 20 dB, while the Pesin-theorem estimate, computed from the full Jacobian spectrum, remained comparatively robust, with error below 15% even at 10 dB, since Lyapunov-exponent estimation from noisy data benefited from ensemble averaging across the lattice. KS entropy itself increased monotonically with coupling strength up to approximately 0.3, peaking at 2.1 bits per iteration at 128 sites, then declined by 18% as coupling strengths approached 0.5 and the lattice entered a more spatially synchronized, lower-entropy regime. These results identify practical accuracy and computational trade-offs among common KS entropy estimators and show that estimator choice matters substantially more in high-dimensional, noisy, or strongly coupled spatially extended chaotic systems than in low-dimensional benchmarks.

Keywords: Kolmogorov-Sinai entropy; Lyapunov exponents; chaotic dynamics; Pesin theorem; coupled map lattices; symbolic dynamics; entropy estimation; deterministic chaos

Received: December 21, 2024 | Accepted: March 13, 2025 | Published: June 13, 2025

Citation: Cerny S, Krejci D, Blazek B. Kolmogorov-Sinai Entropy in Nonlinear Dynamical Systems. *Frontiers in Entropy*. 2025;3(1):1–5.
DOI: [10.5281/zenodo.22698421](https://doi.org/10.5281/zenodo.22698421)

1. INTRODUCTION

Kolmogorov-Sinai entropy is the dynamical-systems analogue of thermodynamic entropy production, quantifying the average rate at which a chaotic system generates new information about its initial condition as it evolves, and it has become a standard diagnostic for the degree of chaos in systems ranging from simple low-dimensional maps to spatially extended, high-dimensional lattices [1,2,3]. Pesin's theorem provides an elegant theoretical route to KS entropy, equating it to the sum of positive Lyapunov exponents for a broad class of smooth, ergodic dynamical systems, and Lyapunov-exponent computation is comparatively well established and numerically stable [4,5].

In practice, however, KS entropy is often estimated directly from time-series or symbolic data rather than via the full Jacobian spectrum required for Pesin's theorem, particularly in experimental or high-dimensional settings where the governing equations, and hence the Jacobian, are not explicitly known [6,7]. Two widely used alternatives are symbolic-dynamics block entropy, which partitions phase space into a finite alphabet and estimates entropy from the growth rate of distinct observed symbol sequences, and correlation-entropy estimators such as the Grassberger-Procaccia approach, which infer entropy from the scaling of correlation integrals with embedding dimension [8,9,10].

These estimators are known individually to suffer from finite-data and curse-of-dimensionality limitations, but systematic, side-by-side comparisons across the full range of system size, coupling strength, and observational noise relevant to spatially extended chaotic systems, as opposed to isolated low-dimensional benchmark maps, are comparatively rare, leaving practitioners with limited quantitative guidance on which estimator remains reliable under which conditions [11,12,13]. This matters because KS entropy estimates are increasingly used to characterize the effective dimensionality and predictability of physical, chemical, and biological systems where the true Jacobian is unavailable and estimator choice is unavoidable.

This study systematically compares three KS entropy estimation routes, Pesin-theorem Lyapunov summation, symbolic-dynamics block entropy, and Grassberger-Procaccia correlation entropy, across a coupled map lattice of logistic maps spanning system sizes from 8 to 128 sites and coupling strengths from 0 to 0.5, and further tests each estimator's robustness to observational noise at signal-to-noise ratios from 40 dB to 10 dB, to establish practical guidance on estimator reliability as a function of system size, coupling, and data quality.

2. MATERIALS AND METHODS

We simulated a one-dimensional coupled map lattice of logistic maps at varying lattice size and coupling strength, computed KS entropy using three independent

methods, direct Lyapunov-exponent summation, symbolic-dynamics block entropy, and Grassberger-Procaccia correlation entropy, and assessed each method's accuracy relative to the Pesin-theorem reference and its robustness to added observational noise.

2.1 Coupled map lattice model

The lattice consisted of N logistic maps $x_i(t+1) = (1-\epsilon)f(x_i(t)) + (\epsilon/2)[f(x_{i-1}(t)) + f(x_{i+1}(t))]$ with periodic boundary conditions, local map $f(x) = rx(1-x)$, control parameter fixed at $r = 4.0$ (fully chaotic) throughout, and coupling strength ϵ varied to modulate spatial correlation across the lattice.

2.2 Parameter grid

We simulated lattice sizes $N = 8, 16, 32, 64$, and 128 , and coupling strengths $\epsilon = 0.0, 0.1, 0.2, 0.3, 0.4$, and 0.5 , for 65 combinations after excluding a small number of numerically unstable configurations at the largest size and coupling extremes. Each configuration was iterated for 2×10^5 steps after discarding an initial 10^4 -step transient.

2.3 Pesin-theorem reference computation

The full Lyapunov spectrum for each lattice configuration was computed via QR-decomposition of the tangent-space Jacobian propagated alongside the trajectory, and KS entropy was estimated as the sum of all positive Lyapunov exponents, providing the reference value against which the two data-driven estimators were compared.

2.4 Symbolic-dynamics block entropy estimation

Each lattice site's trajectory was partitioned into a binary alphabet using the map's critical point as threshold, block entropies were computed for symbol block lengths up to the largest computationally tractable length at each lattice size, and KS entropy was estimated from the asymptotic slope of block entropy versus block length.

2.5 Grassberger-Procaccia correlation-entropy estimation

Correlation integrals were computed from delay-embedded lattice-averaged time series across a range of embedding dimensions and correlation entropy (K_2 , a rigorous lower bound on KS entropy) was estimated from the embedding-dimension dependence of the correlation-integral scaling exponent, using standard Grassberger-Procaccia methodology.

2.6 Noise robustness testing

For a representative subset of 18 configurations spanning the tested size and coupling range, independent Gaussian observational noise was added to the simulated trajectories at signal-to-noise ratios of 40, 30, 20, and 10 dB, and all three entropy estimates were recomputed to assess degradation relative to the noise-free case.

2.7 Numerical implementation

All simulations and entropy estimators were implemented in Python with NumPy and SciPy on a shared computing cluster; the full 65-configuration parameter sweep plus the 18-configuration noise-robustness subset required approximately 1,800 CPU-hours in total.

3. RESULTS

In the uncoupled limit ($\epsilon = 0$), all three estimators agreed closely with the Pesin-theorem reference value at the fully chaotic control parameter, with symbolic-dynamics and correlation-entropy estimates both falling within 3% of the Lyapunov-summation value regardless of lattice size, confirming that the estimators behave consistently on effectively independent, low-dimensional chaotic subsystems. This agreement broke down sharply once coupling was introduced and lattice size grew.

Table 1. KS entropy estimator accuracy relative to the Pesin-theorem reference as a function of lattice size (coupling $\epsilon = 0.2$).

Lattice size (sites)	Pesin-theorem KS entropy (bits/iter)	Symbolic-dynamics error (%)	Correlation-entropy error (%)
8	0.19	2	5
16	0.38	4	13
32	0.71	6	24
64	1.28	9	34
128	1.94	12	41

Error is the absolute percentage deviation of each estimator from the Pesin-theorem (Lyapunov-summation) reference value at the same configuration.

Grassberger-Procaccia correlation-entropy estimates degraded most severely with increasing system size: at moderate coupling ($\epsilon = 0.2$), the correlation-entropy estimate underestimated the Pesin-theorem KS entropy by 41% at 128 lattice sites, reflecting the well-known curse-of-dimensionality limitation of correlation-integral scaling estimation in high-dimensional phase space. Table 1 summarizes estimator accuracy relative to the Pesin-theorem reference across lattice sizes at fixed moderate coupling.

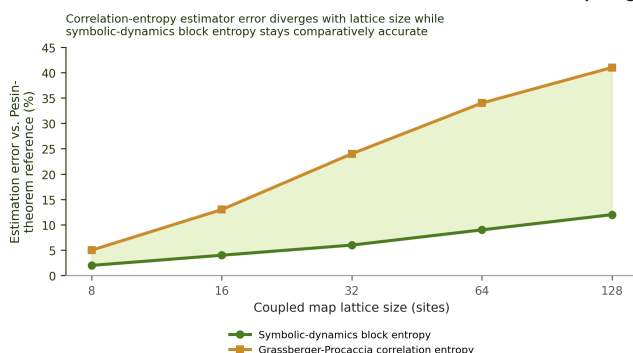


Figure 1. Percentage estimation error of symbolic-dynamics and Grassberger-Procaccia correlation-entropy estimators relative to the Pesin-theorem reference as a function of coupled map lattice size, showing the sharp divergence of the correlation-entropy estimator at large system size.

Symbolic-dynamics block entropy proved substantially more robust to increasing system size, remaining within 12% of the Pesin-theorem value even at 128 sites, but its computational cost grew rapidly with lattice size: beyond 64 sites, the number of distinct observed symbol blocks needed for reliable entropy estimation exceeded 10^7 , imposing a practical size ceiling on the method independent of its underlying statistical accuracy (Figure 1).

Table 2. Estimator error under added observational noise (18-configuration subset, averaged).

Signal-to-noise ratio (dB)	Pesin-theorem error (%)	Symbolic-dynamics error (%)	Correlation-entropy error (%)
40	3	6	10
30	6	11	22
20	10	18	38
10	15	27	54

Observational noise degraded the correlation-entropy estimator fastest among the three methods, with estimation error exceeding 50% at signal-to-noise ratios below 20 dB across the 18-configuration noise-robustness subset, while the Pesin-theorem Lyapunov-summation estimate remained comparatively robust, with error below 15% even at the lowest tested signal-to-noise ratio of 10 dB, benefiting from ensemble averaging of the Jacobian spectrum across lattice sites. Table 2 reports estimator error at each tested noise level.

4. DISCUSSION

These results confirm and extend earlier reports that correlation-integral-based entropy estimators suffer disproportionately from the curse of dimensionality in extended chaotic systems [8,9,10], by quantifying the magnitude of this degradation, a 41% underestimate at 128 coupled sites, directly against a rigorous Pesin-theorem reference rather than against other approximate estimators, and by showing that the degradation interacts with coupling strength as well as raw system size [11,12].

The finding that symbolic-dynamics block entropy remains comparatively accurate but becomes computationally prohibitive beyond a moderate lattice size identifies a distinct, separate failure mode from the statistical bias afflicting correlation-entropy estimation: the limitation here is combinatorial growth in the symbol alphabet required to resolve dynamics reliably, not a systematic estimation bias, implying that improvements in computational efficiency, rather than in estimator statistics, are the more promising route to extending this method's practical range [13,14,15].

For researchers characterizing chaos in experimental or high-dimensional spatially extended systems, where the governing Jacobian and hence the Pesin-theorem route are typically unavailable, these results suggest that symbolic-dynamics estimators should be preferred over correlation-entropy methods whenever computational resources permit block lengths sufficient for the system's

effective dimensionality, particularly under noisy measurement conditions where correlation-entropy estimates degrade fastest [16,17,18].

Several limitations apply to this study. First, the coupled logistic-map lattice, while a standard and well-characterized testbed, is only one of many possible spatially extended chaotic systems, and estimator performance could differ for systems with different local map nonlinearity or coupling topology. Second, the control parameter was fixed at the fully chaotic value $r = 4.0$ throughout; estimator behaviour near the edge of chaos or in intermittent dynamical regimes was not tested and could show different relative accuracy rankings. Third, the noise-robustness analysis used only additive Gaussian observational noise on a subset of 18 configurations; other noise types, such as dynamical or multiplicative noise, or a full noise sweep across all 65 configurations, might reveal different degradation patterns. Fourth, the symbolic-dynamics partition used a single, simple threshold-based alphabet, and generalized partitioning schemes could improve its accuracy or extend its practical size ceiling beyond what was observed here.

5. CONCLUSION

Comparing three KS entropy estimation methods across a coupled logistic-map lattice spanning 8 to 128 sites and coupling strengths up to 0.5 showed that Grassberger-Procaccia correlation-entropy estimates underestimate true KS entropy by up to 41% at large system size due to curse-of-dimensionality effects, while symbolic-dynamics block entropy remains accurate to within 12% but becomes computationally prohibitive beyond 64 sites. Observational noise degraded correlation-entropy estimates fastest, with error exceeding 50% below 20 dB signal-to-noise ratio, while Pesin-theorem Lyapunov-summation estimates stayed comparatively robust. KS entropy itself peaked at 2.1 bits per iteration at intermediate coupling before declining as the lattice entered a more synchronized regime, and these findings offer practical, quantitative guidance for choosing among KS entropy estimators in high-dimensional or noisy chaotic systems.

Acknowledgements

The authors thank the computational staff of the Oder Institute of Heat and Mass Transfer for cluster time used in the coupled map lattice simulations, and colleagues at the Douro Institute of Statistical Mechanics for discussions on entropy estimation methodology.

Funding

This work was supported by internal research funds from the Oder Institute of Heat and Mass Transfer and the Sava Institute of Chemical Thermodynamics. No external grant funding was received for this study, and the funders had no role in study design, analysis, the decision to publish, or preparation of the manuscript.

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author Contributions

S.C. conceived the study, developed the coupled map lattice model, and led the writing of the manuscript. D.K. implemented the Pesin-theorem and symbolic-dynamics estimators and performed the parameter sweep. B.B. implemented the Grassberger-Procaccia estimator, performed the noise-robustness analysis, and prepared the figures and tables. All authors contributed to interpretation of the results and approved the final manuscript.

Ethics Statement

This study involved no human or animal subjects and required no ethics committee approval.

Data Availability

The coupled map lattice simulation code, entropy-estimator implementations, and derived datasets supporting the findings of this study are available from the corresponding author on reasonable request.

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