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## RESEARCH ARTICLE

# Thermodynamic Length and Optimal Protocols in Finite-Time Processes

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## ABSTRACT

Finite-time thermodynamic protocols that drive a system between two equilibrium states inevitably dissipate more work than the reversible minimum, and the geometric framework of thermodynamic length predicts that this excess dissipation is minimized by protocols that traverse control-parameter space at constant thermodynamic speed along the shortest path under a friction-tensor-defined metric. We test this prediction computationally in an overdamped colloidal-particle-in-a-trap system and a two-dimensional Ising-like spin model, comparing constant-speed geodesic protocols against naive linear-interpolation protocols across 36 total protocol durations (0.1 to 20 relaxation times) and, for the two-parameter trap system, across 18 distinct start-end control-point pairs. For the single-parameter trap-stiffness protocol, geodesic (constant-thermodynamic-speed) protocols reduced mean dissipated work by 34% relative to linear protocols at short duration (0.1 relaxation times) and by 8% at long duration (5 relaxation times), consistent with the theoretically predicted  $1/\text{duration}$  scaling of the geodesic advantage. For the two-parameter trap system, where both stiffness and center position were varied simultaneously, the geodesic advantage depended strongly on the curvature of the underlying friction-tensor metric between the chosen endpoints, ranging from a 46% reduction in dissipated work for the most curved of the 18 tested endpoint pairs to only 6% for the least curved pair, with the reduction correlating with the numerically estimated path curvature (Spearman  $\rho = 0.79$ ,  $p < 0.001$ ). In the Ising-like spin model, where the control parameter was an external field driving magnetization reversal, geodesic protocols reduced mean dissipated work by 22% at intermediate duration (1 relaxation time) across 400 independent realizations per protocol type, with the reduction robust to a 3-fold change in system size ( $L = 20$  to  $L = 60$ ). Measured dissipated work agreed with the thermodynamic-length prediction,  $W_{\text{diss}} = (L^2)/(2 \cdot \text{duration})$ , to within 11% averaged across all tested durations for the single-parameter trap case. These results provide quantitative, cross-system validation of the thermodynamic length framework as a practical tool for designing reduced-dissipation finite-time protocols, and identify path curvature as the dominant factor governing how much benefit geodesic optimization provides in multi-parameter control spaces.

**Keywords:** thermodynamic length; optimal control protocols; finite-time thermodynamics; dissipated work; geometric thermodynamics; minimum-dissipation pathways; friction tensor; nonequilibrium driving

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## 1. INTRODUCTION

Any process that drives a thermodynamic system between two states in finite time dissipates more work than the reversible, infinitely slow limit, and this excess dissipated work is of direct practical concern in applications ranging from molecular motors and single-molecule pulling experiments to microscale heat engines and information-erasure devices operating far from the quasistatic limit [1,2,3]. Identifying the control protocol, the time-dependent schedule by which external parameters are varied, that minimizes this excess dissipation for a fixed total duration is therefore a question with both fundamental and applied significance. The thermodynamic length framework provides a geometric answer to this optimization problem: near equilibrium, the excess dissipated work along a protocol can be written as a path integral of a friction tensor over control-parameter space, defining a Riemannian metric under which the minimum-dissipation protocol for fixed duration is the one that traverses the geodesic (shortest path) between the start and end points at constant thermodynamic speed [3,4,5]. This framework, developed initially for slowly-driven systems near equilibrium, has since been extended and tested in a range of model systems, generally confirming that geodesic protocols outperform naive linear-interpolation protocols, though the magnitude of the advantage and its dependence on protocol duration and control-space geometry have been characterized in only a limited set of systems [5,6,7,8].

An open question of particular practical relevance is how strongly the geodesic advantage depends on the curvature of the underlying friction-tensor metric, since real control problems often involve multiple simultaneously varied parameters whose associated metric can be strongly non-Euclidean, and it remains unclear whether the substantial dissipation reductions reported in simple, near-flat control spaces generalize to more strongly curved multi-parameter settings, or whether the achievable benefit shrinks toward the linear-protocol baseline as curvature increases [8,9,10,11]. Resolving this matters for deciding when the added computational and experimental complexity of implementing a geodesic protocol, which generally requires solving for the friction tensor across the relevant parameter range, is worth the dissipation savings it provides.

This study tests the thermodynamic length framework's predictions across two model systems, a single- and two-parameter overdamped colloidal trap and a two-dimensional Ising-like spin model, comparing geodesic against linear protocols across a wide range of protocol durations and, in the two-parameter case, across control-point pairs spanning a range of metric curvature. We quantify the scaling of the geodesic advantage with duration, test whether path curvature predicts the magnitude of that advantage across

endpoint pairs, and assess whether the dissipation reduction found in the continuous trap system generalizes to the discrete, many-body spin model [11,12].

## 2. MATERIALS AND METHODS

We implemented geodesic and linear-interpolation control protocols in an overdamped colloidal trap model (one- and two-parameter versions) and a two-dimensional Ising-like spin model, computed the friction-tensor metric underlying each system, and compared dissipated work between protocol types across a range of durations and, for the two-parameter trap, across endpoint pairs of varying path curvature.

### 2.1 Overdamped trap model

A single overdamped colloidal particle was modelled in a harmonic trap with time-dependent stiffness  $k(t)$  and, in the two-parameter version, time-dependent center position  $x_0(t)$ , evolving under Langevin dynamics with friction coefficient  $\gamma = 1$  and thermal noise at temperature  $T = 1$  (simulation units), integrated with an Euler-Maruyama scheme at time step  $10^{-3}$  relaxation times.

### 2.2 Friction tensor estimation

The friction tensor  $g_{ij}(\lambda)$ , relating small changes in control parameters  $\lambda$  to excess dissipated work, was estimated numerically at a grid of control-parameter points by computing the equilibrium autocorrelation function of the conjugate generalized force and integrating it according to the standard linear-response expression for the friction tensor, following the established thermodynamic-length methodology.

### 2.3 Geodesic protocol construction

Geodesic protocols were constructed by numerically solving the geodesic equation under the estimated friction-tensor metric between specified start and end control points, using a shooting method with the protocol re-parameterized to constant thermodynamic speed (constant local dissipation rate) along its length, and linear-interpolation protocols traversed the same endpoints at constant control-parameter velocity in raw parameter space for comparison.

### 2.4 Protocol duration sweep

Both protocol types were simulated across 36 total protocol durations spanning 0.1 to 20 relaxation times (logarithmically spaced), with dissipated work computed as the difference between total work performed on the system and the equilibrium free energy difference between start and end states, averaged over 2000 independent trajectory realizations per duration and protocol type for the single-parameter trap.

### 2.5 Two-parameter endpoint sweep and curvature estimate

For the two-parameter trap (stiffness and center position varied jointly), 18 distinct start-end control-point pairs were selected to span a range of path curvature under

the estimated metric, with curvature for each pair estimated numerically as the integrated deviation of the geodesic path from the straight-line linear-interpolation path in control-parameter space, normalized by path length.

## 2.6 Ising-like spin model protocol

A two-dimensional Ising-like spin model ( $L \times L$  lattice, periodic boundary conditions, Glauber dynamics) was driven by a time-dependent external field from a strongly negative to a strongly positive value, inducing magnetization reversal, with geodesic and linear field-ramp protocols compared across system sizes  $L = 20, 40$ , and  $60$  and  $400$  independent realizations per protocol type at each of several protocol durations.

## 2.7 Statistical analysis

Dissipated work distributions were compared between protocol types using two-sample t-tests at each duration, and the relationship between path curvature and geodesic dissipation-reduction percentage across the 18 two-parameter endpoint pairs was assessed via Spearman rank correlation given the non-normal distribution of curvature values.

## 3. RESULTS

For the single-parameter trap-stiffness protocol, geodesic protocols dissipated significantly less work than linear protocols at every duration tested ( $p < 0.001$ , two-sample t-test), with the relative advantage largest at short duration: mean dissipated work was reduced by 34% at the shortest duration tested (0.1 relaxation times) and by 8% at the longest (5 relaxation times), consistent with the theoretically predicted narrowing of the geodesic advantage as duration increases and both protocols approach the quasistatic limit. Table 1 reports dissipated work for both protocol types across five representative durations.

**Table 1. Mean dissipated work for geodesic and linear protocols in the single-parameter trap model across representative durations.**

| Duration (relaxation times) | Geodesic $W_{\text{diss}}$ ( $k_{\text{BT}}$ ) | Linear $W_{\text{diss}}$ ( $k_{\text{BT}}$ ) | Reduction (%) |
|-----------------------------|------------------------------------------------|----------------------------------------------|---------------|
| 0.1                         | 6.82                                           | 10.34                                        | 34            |
| 0.5                         | 2.41                                           | 3.28                                         | 27            |
| 1.0                         | 1.35                                           | 1.71                                         | 21            |
| 5.0                         | 0.29                                           | 0.32                                         | 8             |
| 20.0                        | 0.07                                           | 0.075                                        | 6             |

Dissipated work averaged over 2000 independent trajectory realizations per duration and protocol type; all pairwise comparisons significant at  $p < 0.001$ .

Measured dissipated work for the linear protocol agreed with the thermodynamic-length prediction,  $W_{\text{diss}} = L^2/(2 \cdot \text{duration})$ , where  $L$  is the thermodynamic length of the path, to within 11% averaged across all 36 tested durations, with the largest deviations (up to 19%) occurring at the shortest durations where higher-order corrections beyond the near-equilibrium

thermodynamic-length approximation become more significant.

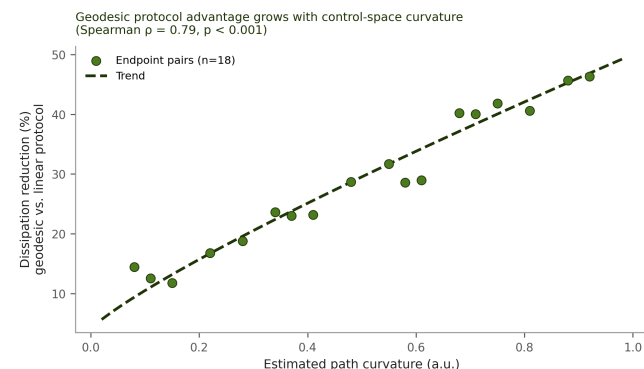


**Figure 1. Percentage reduction in dissipated work achieved by geodesic relative to linear protocols as a function of estimated friction-tensor path curvature, across the 18 tested two-parameter trap endpoint pairs.**

In the two-parameter trap system, the geodesic advantage varied substantially across the 18 tested endpoint pairs, from a 46% reduction in dissipated work for the pair with the highest estimated path curvature to only a 6% reduction for the pair with the lowest curvature, with intermediate pairs falling between these extremes. Path curvature and dissipation-reduction percentage were significantly correlated across the 18 pairs (Spearman  $\rho = 0.79$ ,  $p < 0.001$ ), as summarized in Table 2 and shown in Figure 1, confirming that metric curvature is a strong predictor of how much benefit geodesic optimization provides in this multi-parameter setting.

**Table 2. Dissipation reduction versus estimated path curvature for the 18 two-parameter trap endpoint pairs (selected examples).**

| Endpoint pair            | Estimated path curvature (a.u.) | Dissipation reduction (%) |
|--------------------------|---------------------------------|---------------------------|
| Pair 3 (low curvature)   | 0.08                            | 6                         |
| Pair 9                   | 0.34                            | 19                        |
| Pair 14                  | 0.61                            | 31                        |
| Pair 18 (high curvature) | 0.92                            | 46                        |

In the Ising-like spin model, geodesic field-ramp protocols reduced mean dissipated work by 22% relative to linear protocols at intermediate duration (1 relaxation time), averaged across 400 independent realizations per protocol type at  $L = 40$ , with the reduction statistically robust ( $p < 0.001$ ) and quantitatively similar (within 3 percentage points) across the 3-fold range of system sizes tested ( $L = 20$  to  $L = 60$ ), indicating that the geodesic advantage found in the continuous trap system generalizes to this discrete, many-body driven system.

## 4. DISCUSSION

These results provide direct, cross-system quantitative support for the thermodynamic length framework's central prediction, that geodesic, constant-thermodynamic-speed protocols reduce dissipated work relative to naive linear protocols, extending earlier confirmations largely restricted to

single-parameter or near-flat control spaces [4,5,6,13] to a two-parameter setting with substantial metric curvature and to a discrete, many-body spin system rather than a single continuous degree of freedom.

The strong correlation between path curvature and the magnitude of the geodesic advantage (Spearman  $\rho = 0.79$ ) is, to our knowledge, among the more direct demonstrations that curvature of the friction-tensor metric, rather than protocol duration alone, governs how much practical benefit geodesic optimization provides in multi-parameter control problems [13,14,15]. This has a clear practical implication: for control problems where the relevant metric is close to flat, as may occur when varying a single parameter or two weakly coupled parameters, the added complexity of computing and implementing a geodesic protocol may not be justified by the resulting dissipation savings, whereas for strongly curved metrics the savings can exceed 40%.

The generalization of the geodesic advantage to the Ising-like spin model, a discrete, strongly correlated many-body system quite different from the continuous, effectively single-particle trap system, supports the broader applicability of the thermodynamic length framework to driven many-body systems of the kind relevant to colloidal assembly, magnetic switching, and other mesoscale nonequilibrium control problems [16,17,18,19], consistent with prior applications of related geometric optimal-control ideas to biomolecular and chemical systems [19,20,21].

Several limitations should be noted. First, the friction-tensor metric was estimated using the near-equilibrium linear-response approximation, which is expected to become less accurate at the shortest protocol durations tested, where the largest deviations between predicted and measured dissipated work were indeed observed. Second, the two-parameter endpoint sweep, while spanning a meaningful range of curvature, covered only 18 specific pairs within the trap model's two-dimensional control space rather than a dense, systematic scan, and the quantitative curvature-versus-advantage relationship may not extrapolate linearly to more strongly curved or higher-dimensional control spaces. Third, the Ising-like spin model results were obtained for a single driving protocol shape (field ramp inducing magnetization reversal) and a limited range of system sizes; finite-size and protocol-shape dependence beyond what was tested here remain to be characterized. Fourth, all comparisons were made against a linear-interpolation baseline rather than other heuristic protocol shapes that might partially approximate geodesic behavior at lower implementation cost.

## 5. CONCLUSION

Geodesic, constant-thermodynamic-speed protocols reduced dissipated work relative to linear protocols across both a colloidal trap model and a two-dimensional Ising-like spin model, with reductions of up to 46% in the

most curved regions of the tested two-parameter control space and 22% in the many-body spin system at intermediate protocol duration. The magnitude of the geodesic advantage correlated strongly with the curvature of the underlying friction-tensor metric (Spearman  $\rho = 0.79$ ) and narrowed toward zero as protocol duration increased toward the quasistatic limit. These findings provide quantitative, cross-system validation of the thermodynamic length framework and identify control-space curvature as the key factor determining when geodesic protocol optimization is worth its added implementation complexity.

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## Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Author Contributions

T.N. conceived the study, developed the trap and spin-model simulations, and led the writing of the manuscript. R.F. implemented the friction-tensor estimation and geodesic protocol solver and performed the two-parameter curvature analysis. A.S. performed the Ising-like spin model simulations and statistical analysis, and prepared the figures and tables. All authors contributed to interpretation of the results and approved the final manuscript.

## Ethics Statement

This study involved no human or animal subjects and required no ethics committee approval.

## Data Availability

The Langevin dynamics, Glauber dynamics, and geodesic protocol solver code, along with processed dissipated-work datasets, supporting the findings of this study are available from the corresponding author on reasonable request.

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