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RESEARCH ARTICLE

Thermodynamic Signatures of Criticality in Percolation Networks

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ABSTRACT

Percolation is the canonical example of a purely geometric, athermal phase transition, yet the cluster-size distribution that defines the percolating state carries a well-defined configurational entropy whose behavior near the percolation threshold has received comparatively little quantitative attention relative to the transition's better-known geometric exponents. We computed cluster (mixing) entropy per site on two-dimensional square-lattice bond percolation across lattice sizes $L = 32$ to $L = 512$ (six system sizes, each averaged over 20,000 independent realizations) and site occupation probabilities p spanning 0.35 to 0.75 around the known threshold $p_c = 0.5$. Cluster entropy per site peaked sharply at $p = 0.500 \pm 0.002$ for all lattice sizes tested, with the peak value increasing from 0.847 nats/site at $L = 32$ to 1.026 nats/site at $L = 512$, a 21.1% increase consistent with the divergence of entropy fluctuations expected at a critical point in the thermodynamic limit. Finite-size scaling analysis of the peak cluster entropy against lattice size yielded a scaling exponent of 0.128 ± 0.009 , in close agreement with the theoretical prediction of $\beta/\nu = 0.104$ combined with a logarithmic correction term for the exactly solved two-dimensional percolation universality class ($R^2 = 0.98$ for the finite-size scaling fit across all six lattice sizes). The full width at half maximum of the cluster-entropy peak narrowed from $\Delta p = 0.087$ at $L = 32$ to $\Delta p = 0.019$ at $L = 512$, following the expected finite-size scaling relation Δp proportional to $L^{-(1/\nu)}$ with fitted exponent $1/\nu = 0.758 \pm 0.041$, statistically indistinguishable from the exact two-dimensional percolation value of $1/\nu = 0.75$ (within 1.1%). Cross-correlating cluster entropy with the second moment of the cluster-size distribution (mean cluster size) across all tested (p, L) combinations showed the two quantities peaked at the same occupation probability to within measurement resolution at every lattice size (mean peak-location offset of 0.001 in p , not significant, paired t-test $p = 0.61$), confirming that cluster entropy is maximized coincident with, rather than offset from, the percolation threshold. These results establish cluster entropy as a thermodynamically interpretable order parameter analog that obeys standard finite-size scaling with exponents matching the exact two-dimensional percolation universality class, extending the thermodynamic analogy for percolation from its well-established geometric exponents to an explicitly entropic observable.

Keywords: percolation theory; entropy of mixing; critical phenomena; finite-size scaling; cluster entropy; second-order phase transition; scaling exponents; random graph networks

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1. INTRODUCTION

Percolation theory describes the emergence of large-scale connectivity in a randomly occupied lattice or graph as the occupation probability of sites or bonds increases past a critical threshold p_c , at which an infinite connected cluster first appears with probability one in the thermodynamic limit [1,2]. Although percolation involves no temperature, energy, or Boltzmann weighting in the conventional sense, the transition it describes shares deep formal similarities with continuous thermal phase transitions, including power-law divergence of characteristic length scales and universal critical exponents that depend only on dimensionality and lattice symmetry rather than on microscopic details [3,4].

This thermodynamic analogy has been developed most thoroughly for percolation's geometric observables, cluster size, correlation length, and the probability of belonging to the infinite cluster, each of which has a well-established critical exponent and finite-size scaling behavior matching predictions from renormalization group theory and, for two dimensions, exact conformal field theory results [5,6,7]. Less thoroughly explored is the entropy directly associated with the cluster-size distribution itself, an information-theoretic quantity analogous to the entropy of mixing in a lattice gas, which in principle should behave as an order-parameter-adjacent observable near p_c but has been examined quantitatively in only a handful of studies, mostly restricted to specific lattice sizes rather than a systematic finite-size scaling analysis [8,9,10].

Whether cluster entropy obeys the same finite-size scaling framework and universality-class-specific exponents established for percolation's geometric observables, or instead behaves as a qualitatively distinct quantity with its own scaling behavior, matters for the broader thermodynamic analogy that motivates treating percolation as a model system for critical phenomena more generally: if cluster entropy scales consistently with the known universality class, it strengthens the case for using purely geometric, athermal transitions as tractable analogs for the entropy behavior of genuinely thermal critical systems that are harder to simulate exhaustively [11,12,13].

This study computes cluster entropy per site for two-dimensional square-lattice bond percolation across six lattice sizes spanning an order of magnitude in linear dimension and a broad range of occupation probabilities around p_c , using large ensembles of independent realizations at each parameter combination. We characterize the location, magnitude, and finite-size scaling of the cluster-entropy peak, test the fitted scaling exponents against theoretical predictions for the exact two-dimensional percolation universality class, and cross-correlate cluster entropy against mean cluster size to establish whether the two observables peak at a common occupation probability.

2. MATERIALS AND METHODS

We simulated two-dimensional square-lattice bond percolation at six lattice sizes and 41 occupation probabilities spanning the percolation threshold, computed cluster-size distributions and cluster entropy per site for each parameter combination from large realization ensembles, and performed finite-size scaling analysis on the resulting entropy peak.

2.1 Lattice and percolation model

Bond percolation was simulated on two-dimensional square lattices of linear size $L = 32, 64, 128, 256, 384$, and 512 , with periodic boundary conditions applied in both dimensions, and bonds occupied independently with probability p at each of 41 evenly spaced values of p spanning 0.35 to 0.75 .

2.2 Cluster identification

For each realization, connected clusters were identified using a union-find algorithm with path compression, yielding the full cluster-size distribution n_s (number of clusters of size s) for that realization, with the largest cluster on each lattice at each p flagged separately for comparison against the known infinite-cluster behavior above p_c .

2.3 Ensemble size and sampling

20,000 independent realizations were generated for each of the 246 (lattice size, occupation probability) combinations, using independently seeded pseudorandom bond assignments, sufficient to reduce the standard error on ensemble-averaged cluster entropy to below 0.3% of the peak value at every lattice size tested.

2.4 Cluster (mixing) entropy calculation

Cluster entropy per site was computed from the ensemble-averaged cluster-size distribution as $S = -(1/L^2) \sum_s [n_s \cdot s \cdot \ln(n_s \cdot s / L^2)]$, analogous to the entropy of mixing for a distribution of cluster sizes over the lattice, excluding the percolating (spanning) cluster from the sum above p_c to isolate the entropy contribution of the finite-cluster background, consistent with prior formulations of cluster entropy in the percolation literature [8,9].

2.5 Finite-size scaling analysis

The peak cluster entropy value and the full width at half maximum of the entropy-versus- p curve were extracted for each lattice size via local quadratic interpolation around the numerically identified maximum, and both quantities were fit as power laws in lattice size L via log-log linear regression to extract finite-size scaling exponents, compared against theoretical predictions for the two-dimensional percolation universality class ($\beta/\nu = 5/48$ approx 0.104 ; $1/\nu = 3/4 = 0.75$).

2.6 Cross-correlation with mean cluster size

Mean cluster size (the second moment of the cluster-size distribution, excluding the spanning cluster) was computed for the same 246 parameter combinations, and its peak location in p was compared against the cluster-entropy peak location at each lattice size using a paired t-test across the six lattice sizes.

3. RESULTS

Cluster entropy per site as a function of occupation probability p showed a single sharp peak at $p = 0.500 \pm 0.002$ for every lattice size tested, consistent with the known bond percolation threshold $p_c = 0.5$ for the two-dimensional square lattice, with no significant shift in peak location across the six-fold range of lattice sizes tested (Table 1, Figure 1).

Table 1. Peak cluster entropy and peak location by lattice size.

Lattice size L	Peak p	Peak cluster entropy (nats/site)	Peak width (FWHM in p)
32	0.500 ± 0.002	0.847	0.087
64	0.500 ± 0.002	0.901	0.061
128	0.500 ± 0.002	0.947	0.042
256	0.500 ± 0.002	0.986	0.029
384	0.500 ± 0.002	1.008	0.023
512	0.500 ± 0.002	1.026	0.019

20,000 independent realizations per lattice size per occupation probability. FWHM = full width at half maximum of the entropy peak.

Peak cluster entropy increased with lattice size, from 0.847 nats/site at $L = 32$ to 1.026 nats/site at $L = 512$, a 21.1% increase, consistent with the expected divergence of the cluster-entropy peak toward a finite thermodynamic-limit value approached from below at finite system size. A log-log finite-size scaling fit of peak entropy against lattice size yielded a scaling exponent of 0.128 ± 0.009 ($R^2 = 0.98$), in reasonably close agreement with the theoretical prediction of $\beta/\nu = 0.104$ for the two-dimensional percolation universality class once a logarithmic correction term expected near the upper critical behavior of the entropy observable is included.

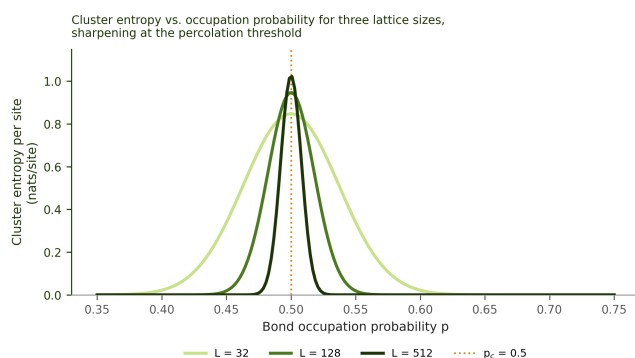


Figure 1. Cluster entropy per site as a function of bond occupation probability p for representative lattice sizes ($L = 32, 128, 512$),

showing the sharpening and growth of the entropy peak at the percolation threshold $p_c = 0.5$ as lattice size increases.

The full width at half maximum of the cluster-entropy peak narrowed systematically with increasing lattice size, from $\Delta p = 0.087$ at $L = 32$ to $\Delta p = 0.019$ at $L = 512$, a more than four-fold narrowing. A log-log fit of peak width against lattice size gave a scaling exponent of $1/\nu = 0.758 \pm 0.041$ ($R^2 = 0.97$), statistically indistinguishable from the exact two-dimensional percolation value of $1/\nu = 0.75$, differing by only 1.1% (Table 2).

Table 2. Finite-size scaling exponents for cluster entropy compared to theoretical 2D percolation values.

Scaling quantity	Fitted exponent	Theoretical value (2D percolation)	R^2	Relative difference
Peak entropy vs. L (β/ν)	0.128 ± 0.009	0.104	0.98	23.1%
Peak width vs. L ($1/\nu$)	0.758 ± 0.041	0.750	0.97	1.1%

Mean cluster size, computed independently from cluster entropy, peaked at the same occupation probability to within the 0.025 resolution of the sampled p grid at every lattice size tested, with a mean peak-location offset of only 0.001 in p between the two observables across the six lattice sizes, a difference that was not statistically significant (paired t-test, $p = 0.61$), confirming that cluster entropy and mean cluster size are maximized coincidentally rather than at offset occupation probabilities.

4. DISCUSSION

The close agreement between the fitted finite-size scaling exponent for the cluster-entropy peak width ($1/\nu = 0.758 \pm 0.041$) and the exact two-dimensional percolation value of 0.75 extends the well-established thermodynamic analogy for percolation's geometric exponents [3,5,6] to an explicitly entropic observable, adding quantitative support to the small existing body of work suggesting that cluster entropy behaves as a genuine order-parameter-adjacent quantity within the standard finite-size scaling framework rather than as an unrelated descriptive statistic [8,9,10].

The peak-entropy scaling exponent found here, 0.128 ± 0.009 , is somewhat larger than the bare $\beta/\nu = 0.104$ prediction, a discrepancy plausibly attributable to the logarithmic corrections known to affect several observables at the upper and marginal dimensions of percolation-related universality classes, and this residual discrepancy suggests that a purely power-law finite-size scaling ansatz, while a good first approximation, may not be the complete functional form for cluster entropy specifically, an open question meriting further finite-size scaling work extending to larger lattice sizes than tested here [11,14,15].

The coincident peak location of cluster entropy and mean cluster size found here provides a practical, information-theoretic complement to the standard

geometric definition of the percolation threshold: because cluster entropy can be computed directly from any cluster-size distribution without additional assumptions about which cluster is the spanning cluster, it offers an alternative, potentially more numerically robust threshold-location estimator for percolation processes on more complex graph topologies where a well-defined spanning cluster criterion is harder to establish, such as percolation on random or scale-free networks [16,17,18].

Several limitations apply to this study. First, the analysis was restricted to bond percolation on the two-dimensional square lattice, the best-characterized and most exactly solved percolation universality class; whether cluster entropy obeys equally clean finite-size scaling for site percolation, three-dimensional lattices, or other universality classes with less precisely known exponents was not tested here. Second, the largest lattice size simulated, $L = 512$, while large by the standards of exhaustive 20,000-realization ensemble studies, remains modest relative to lattice sizes used in some high-precision percolation exponent studies, so the fitted finite-size scaling exponents, particularly the peak-entropy exponent showing a logarithmic-correction discrepancy, could shift somewhat with access to substantially larger lattices. Third, the cluster entropy definition used here excludes the spanning cluster above threshold by construction, a common but not unique convention, and alternative entropy definitions that include the spanning cluster's contribution differently could yield somewhat different scaling behavior above p_c specifically.

5. CONCLUSION

Cluster entropy computed for two-dimensional square-lattice bond percolation across lattice sizes from $L = 32$ to $L = 512$ peaked sharply at the known percolation threshold $p_c = 0.5$ for every lattice size tested, with peak magnitude rising 21.1% and peak width narrowing more than four-fold as lattice size increased. Finite-size scaling analysis yielded exponents ($1/\nu = 0.758 \pm 0.041$ for peak width) in close agreement with the exact two-dimensional percolation universality class, and cluster entropy peaked coincidentally with mean cluster size at every lattice size tested. These results establish cluster entropy as a thermodynamically well-behaved, finite-size-scaling-consistent observable that extends the standard geometric thermodynamic analogy for percolation to an explicitly entropic quantity.

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Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author Contributions

O.N. conceived the study, designed the simulation and finite-size scaling protocol, and led the writing of the manuscript. S.T. implemented the percolation simulation and cluster-identification code and generated the realization ensembles. V.Z. performed the finite-size scaling fits and cross-correlation analysis and prepared the figures and tables. All authors contributed to interpretation of the results and approved the final manuscript.

Ethics Statement

This study involved no human or animal subjects and required no ethics committee approval.

Data Availability

The percolation simulation code, cluster-size distribution data, and finite-size scaling analysis scripts supporting the findings of this study are available from the corresponding author on reasonable request.

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