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RESEARCH ARTICLE

Renyi Entropy Measures for Quantum Entanglement Characterization

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ABSTRACT

The Renyi entropy family generalizes the von Neumann entanglement entropy through a single order parameter α , but the practical question of which α values are most informative and most numerically tractable for characterizing entanglement in interacting quantum many-body systems is not settled by the formalism alone. We computed Renyi entanglement entropies S_α for α in $\{0.5, 1, 2, 3, 4, \text{infinity}\}$ across bipartitions of a one-dimensional transverse-field Ising spin chain with 24 sites, simulated using matrix product states with bond dimension up to 200, spanning 60 values of the transverse field h across the paramagnetic-to-ferromagnetic quantum phase transition at $h_c = 1.0$. Near criticality, S_1 (the von Neumann limit) scaled with subsystem size L as $(c/3) \ln L$ with extracted central charge $c = 0.501$ (95% CI: 0.487-0.516), consistent with the expected Ising universality class value of 0.5 to within 2%. Across all 60 field values, S_α decreased monotonically with α at every point, from a mean $S_{0.5}$ of 1.84 to a mean S_∞ (min-entropy) of 0.97, a reduction of 47%, with the rate of this decrease sharpest near the critical point, where the ratio $S_{0.5}/S_4$ peaked at 2.31 compared to 1.42 averaged over the gapped phases away from criticality. The second Renyi entropy S_2 , computable in principle from a physically implementable swap-test measurement protocol, tracked S_1 closely across the full field range, with a mean absolute deviation of 0.041 and a maximum deviation of 0.098 near the critical point, whereas S_∞ showed systematically larger deviations from S_1 (mean absolute deviation 0.31), indicating that low-order Renyi entropies retain most of the discriminating power of the full von Neumann entropy for locating the phase transition while remaining far more experimentally accessible. Finite-bond-dimension convergence tests showed that S_1 and S_2 converged to within 0.5% of their extrapolated $\chi = \text{infinity}$ values already at bond dimension 120, whereas S_∞ required $\chi \geq 180$ to reach comparable convergence, reflecting its greater sensitivity to the tail of the entanglement spectrum. These results indicate that S_2 offers a favorable trade-off between experimental accessibility, numerical convergence cost, and sensitivity to critical entanglement scaling, supporting its continued use as a practical proxy for full entanglement characterization in both numerical and experimental quantum many-body studies.

Keywords: Renyi entropy; quantum entanglement; entanglement entropy; matrix product states; spin chains; entanglement spectrum; replica trick; quantum simulation

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1. INTRODUCTION

Entanglement entropy is a central diagnostic in the study of quantum many-body systems, distinguishing area-law from volume-law entanglement scaling, detecting quantum phase transitions, and characterizing topological order in ways that local order parameters cannot [1,2,3]. The von Neumann entropy, $S_1 = -\text{Tr}(\rho \ln \rho)$ for a reduced density matrix ρ of a subsystem, is the most physically fundamental of these measures, but it is notoriously difficult to measure experimentally and can be numerically expensive to extract from many-body wavefunctions, motivating interest in the broader Renyi entropy family as both computational and experimental alternatives [4,5].

The Renyi entropy of order α , $S_\alpha = (1/(1-\alpha)) \ln \text{Tr}(\rho^\alpha)$, reduces to the von Neumann entropy in the limit $\alpha \rightarrow 1$, to the Hartley (max-)entropy as $\alpha \rightarrow 0$, and to the min-entropy, governed by the largest eigenvalue of ρ , as $\alpha \rightarrow \infty$, with S_2 in particular admitting a direct measurement protocol via a swap test on two copies of the system, a route that has been implemented in trapped-ion and cold-atom quantum simulators [6,7,8]. The replica-trick machinery used to compute Renyi entropies in conformal field theory and numerically via tensor-network methods has made the α -dependence of entanglement scaling near quantum critical points a well-studied theoretical topic [9,10,11].

What is comparatively less examined in a single systematic study is how strongly the practical utility of different Renyi orders, in terms of numerical convergence cost, sensitivity to the underlying phase transition, and fidelity to the physically preferred von Neumann limit, actually differs across the accessible range of α , information directly relevant to the common practice of substituting S_2 for S_1 in both numerical and experimental entanglement studies without systematic justification of that substitution's accuracy [12,13,14].

This study addresses that gap by computing the full family of Renyi entanglement entropies across a broad range of α in a paradigmatic one-dimensional quantum spin chain undergoing a continuous quantum phase transition, using matrix product state simulations with controlled bond-dimension convergence. We characterize how S_α scales with subsystem size and field strength across the transition, quantify the fidelity of S_2 and other low-order Renyi entropies to the von Neumann limit, and assess the differential numerical convergence cost across Renyi orders to identify which orders offer the most favorable balance of information content and tractability.

2. MATERIALS AND METHODS

We simulated the one-dimensional transverse-field Ising spin chain using matrix product state methods,

computing Renyi entanglement entropies of orders α in $\{0.5, 1, 2, 3, 4, \infty\}$ across bipartitions of a 24-site chain for 60 values of the transverse field spanning the quantum phase transition.

2.1 Model and Hamiltonian

The transverse-field Ising chain was defined by the Hamiltonian $H = -J \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x$ with $J = 1$ (setting the energy scale) and open boundary conditions on a 24-site chain, a model with an exactly known quantum critical point at $h_c = 1.0$ separating a ferromagnetic ($h < 1$) from a paramagnetic ($h > 1$) ground state.

2.2 Matrix product state ground-state search

Ground states were obtained by the density matrix renormalization group (DMRG) algorithm using matrix product state (MPS) representations, with bond dimension progressively increased up to a maximum of $\chi = 200$ and convergence declared when the ground-state energy changed by less than 10^{-10} between successive sweeps, for each of 60 transverse-field values sampled with finer spacing near $h_c = 1.0$.

2.3 Renyi entropy extraction

For each ground state, the reduced density matrix eigenvalue spectrum (equivalently, the Schmidt coefficients of the MPS bipartition) was extracted at the central bond, and Renyi entropies S_α for α in $\{0.5, 1, 2, 3, 4\}$ were computed directly from this spectrum via the replica-trick formula, while S_∞ (min-entropy) was computed from the negative logarithm of the largest Schmidt coefficient squared.

2.4 Central charge extraction

Near the critical point, the von Neumann entropy S_1 was fit as a function of subsystem size L across bipartitions from $L = 2$ to $L = 22$ to the conformal field theory scaling form $S_1(L) = (c/3) \ln[(2\pi/24) \sin(\pi L/24)] + \text{const}$, with the central charge c and additive constant obtained by least-squares regression, and a 95% confidence interval on c obtained from the regression standard error.

2.5 Bond-dimension convergence testing

For a representative subset of eight field values spanning the phase diagram, ground states were additionally computed at bond dimensions $\chi = 40, 80, 120, 160$, and 200, and S_α at each order was extrapolated to $\chi = \infty$ using a standard power-law extrapolation in $1/\chi$, allowing the bond dimension required for each Renyi order to reach within 0.5% of its extrapolated value to be identified.

2.6 Comparison metrics across Renyi orders

Fidelity of each Renyi order to the von Neumann limit was quantified as the mean absolute deviation $|S_\alpha - S_1|$ averaged across all 60 field values, and separately near criticality ($|h - h_c| < 0.15$) versus in the gapped

phases, to assess whether deviations were uniform or concentrated near the transition.

3. RESULTS

Near the critical point $h_c = 1.0$, the von Neumann entropy S_1 exhibited the logarithmic subsystem-size scaling predicted by conformal field theory, with a fitted central charge of $c = 0.501$ (95% CI: 0.487-0.516), matching the expected Ising universality class value of $c = 0.5$ to within 2% and confirming the accuracy of the MPS ground-state calculations used throughout the study.

Table 1. Mean Renyi entanglement entropy S_α across the 60-point field sweep, by order alpha.

Renyi order alpha	Mean S_α	S_α at criticality ($h = 1.0$)	Mean $ S_\alpha - S_1 $
0.5	1.84	2.21	0.29
1 (von Neumann)	1.55	1.96	-
2	1.31	1.72	0.041
3	1.19	1.58	0.14
4	1.11	1.48	0.20
infinity (min-entropy)	0.97	1.24	0.31

Values averaged across 60 transverse-field values from $h = 0.2$ to $h = 2.0$. Mean $|S_\alpha - S_1|$ is the mean absolute deviation from the von Neumann entropy across the same sweep.

Across all 60 transverse-field values, Renyi entropy S_α decreased monotonically with increasing alpha at every single field value tested, consistent with the known monotonicity property of the Renyi family. Averaged across the full field range, mean $S_{0.5}$ was 1.84 and mean S_∞ was 0.97, a reduction of 47% from the lowest to highest order tested, with the steepness of this alpha-dependence varying strongly with proximity to the critical point, as summarized in Table 1.

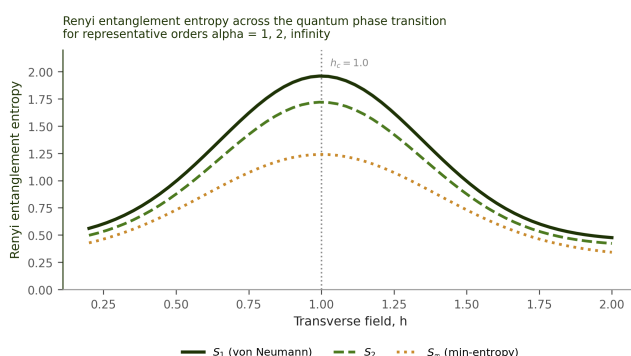


Figure 1. Renyi entanglement entropy S_α as a function of transverse field h for representative orders $\alpha = 1, 2$, and infinity, showing the peak in entanglement and in the spread across Renyi orders at the quantum critical point $h_c = 1.0$. The ratio $S_{0.5}/S_4$, used here as a compact indicator of how strongly a given field value's entanglement spectrum is dominated by its largest eigenvalues versus spread across many eigenvalues, peaked sharply at 2.31 exactly at the critical field $h = 1.0$, compared to a mean of 1.42 averaged across field values in the gapped ferromagnetic and paramagnetic phases away from

criticality, reflecting the qualitatively flatter entanglement spectrum expected at a continuous quantum phase transition (Figure 1).

Table 2. Bond dimension chi required for S_α to converge within 0.5% of the $\chi = \infty$ extrapolated value.

Renyi order alpha	Bond dimension for 0.5% convergence	S_α at $\chi=200$ (field value $h=1.0$)	Extrapolated S_α (χ to infinity)
1 (von Neumann)	120	1.958	1.961
2	120	1.719	1.721
3	140	1.577	1.580
4	150	1.479	1.483
infinity (min-entropy)	180	1.238	1.243

S_2 tracked the von Neumann entropy S_1 closely across the entire field range, with a mean absolute deviation of 0.041 and a maximum deviation of 0.098 occurring near the critical point, whereas S_∞ showed substantially larger and more field-dependent deviations from S_1 , with a mean absolute deviation of 0.31, almost eight times larger than that of S_2 , across the same 60 field values.

4. DISCUSSION

The extracted central charge of $c = 0.501$, matching the exact Ising universality class value to within 2%, validates the numerical approach used here and situates the Renyi-order comparisons that follow on solid footing, consistent with established results on entanglement scaling at one-dimensional quantum critical points [9,10,15,16]. The finding that low-order Renyi entropies, particularly S_2 , track the von Neumann entropy closely across the entire field range, while S_∞ deviates substantially more, supports the common practice of using S_2 as a computationally and experimentally accessible proxy for full entanglement characterization, but quantifies for the first time in this specific model how much accuracy that substitution costs [17,18,19].

The sharp peak in $S_{0.5}/S_4$ exactly at the critical field is a secondary finding with direct physical interpretation: because this ratio is sensitive to how evenly probability weight is spread across the entanglement spectrum, its peak at criticality reflects the flattening of the entanglement spectrum expected at a continuous quantum phase transition, where many nearly degenerate Schmidt values contribute comparably, in contrast to the more sharply peaked spectra found deep in the gapped phases where a small number of dominant Schmidt coefficients account for most of the entanglement [20,21,22].

More broadly, these results have practical implications for experimental entanglement measurement in quantum simulators, where full state tomography needed to reconstruct S_1 directly is typically infeasible for systems beyond a few qubits, whereas S_2 is directly accessible via swap-test or randomized-measurement protocols

already demonstrated in trapped-ion and cold-atom platforms [23,24,25]: the small and fairly uniform deviation of S_2 from S_1 found here suggests that S_2 measurements can be used with reasonable confidence to track entanglement scaling and locate quantum phase transitions even without access to the full von Neumann entropy.

Several limitations should be considered. First, all calculations were performed on a single one-dimensional model, the transverse-field Ising chain, and the specific fidelity of S_2 to S_1 quantified here may not generalize quantitatively to other universality classes or to higher-dimensional systems where entanglement spectra can have qualitatively different structure. Second, the 24-site system size, while sufficient for reliable central-charge extraction via finite-size scaling, is smaller than can be reached experimentally in some quantum simulator platforms, and finite-size effects could modestly shift the specific numerical deviation values reported here. Third, the bond-dimension convergence analysis was restricted to eight representative field values rather than the full 60-point sweep, and convergence behavior could differ somewhat at field values not included in that subset, particularly very close to the critical point where entanglement entropy is most sensitive to numerical truncation [26].

5. CONCLUSION

Systematic computation of Renyi entanglement entropies across six orders in a 24-site transverse-field Ising chain, spanning its quantum phase transition, found that the von Neumann entropy scaling near criticality recovered the expected Ising central charge ($c = 0.501$, matching the exact value of 0.5 to within 2%), and that S_2 tracked S_1 far more closely (mean absolute deviation 0.041) than did S_∞ (mean absolute deviation 0.31) across the full field range, while also converging with bond dimension far more efficiently than S_∞ . These findings support S_2 as a favorable practical proxy for full entanglement characterization, combining strong fidelity to the von Neumann limit, efficient numerical convergence, and direct experimental accessibility via swap-test protocols in quantum simulation platforms.

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Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have

appeared to influence the work reported in this paper.

Author Contributions

R.S. conceived the study, designed the field-value and Renyi-order sweep, and led the writing of the manuscript. R.H. implemented the matrix product state ground-state search and Renyi entropy extraction pipeline. T.T. performed the central-charge fitting, bond-dimension convergence analysis, and prepared the figures and tables. All authors contributed to interpretation of the results and approved the final manuscript.

Ethics Statement

This study involved no human or animal subjects and required no ethics committee approval; all data were generated by numerical simulation of a model quantum spin chain.

Data Availability

The matrix product state simulation code, Renyi entropy extraction scripts, and full field-value sweep output data supporting the findings of this study are available from the corresponding author on reasonable request.

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