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## RESEARCH ARTICLE

## Non-Extensive Statistical Mechanics in Long-Range Interacting Systems

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### ABSTRACT

Systems with long-range interactions, where the pairwise coupling decays as a power law with an exponent smaller than the system's spatial dimension, violate the additivity assumption underlying Boltzmann-Gibbs statistical mechanics and have been proposed as natural candidates for Tsallis's non-extensive  $q$ -entropy framework. We tested this proposal numerically in the Hamiltonian mean-field (HMF) model, simulating  $N$ -body dynamics for system sizes  $N$  from 500 to 8,000 particles with pairwise coupling decaying as  $r^{-\alpha}$  for six values of the range parameter  $\alpha$  between 0 (fully long-range) and 1.5 (short-range-like), following each system's relaxation from a fixed out-of-equilibrium initial condition (water-bag distribution) toward its final equilibrium state. For  $\alpha \leq 0.6$ , all system sizes tested showed an extended quasi-stationary state (QSS) before final relaxation to Boltzmann-Gibbs equilibrium, with QSS lifetime scaling with system size as  $N^{1.68}$  (95% CI: 1.59–1.77) at  $\alpha = 0$ , consistent with the divergent-lifetime signature expected for genuinely long-range coupling, and momentum distributions during the QSS phase were fit substantially better by a  $q$ -Gaussian (Tsallis) distribution than by a Maxwell-Boltzmann distribution, with mean fitted  $q = 1.42$  (SD 0.08) across the six  $N$  values tested at  $\alpha = 0$ . The quality advantage of the  $q$ -Gaussian fit, measured as the reduction in reduced chi-squared relative to the Maxwell-Boltzmann fit, was 68% at  $\alpha = 0$ , fell to 34% at  $\alpha = 0.6$ , and became negligible (4%, not significant at  $p = 0.31$ ) for  $\alpha \geq 1.0$ , tracing a clear crossover from non-extensive to extensive statistical behavior as the interaction range parameter increased. QSS lifetime scaling exponents decreased systematically with  $\alpha$ , from 1.68 at  $\alpha = 0$  to 0.71 at  $\alpha = 0.6$ , and QSS states were no longer statistically distinguishable from rapid Boltzmann-Gibbs relaxation for  $\alpha \geq 1.0$  across all tested  $N$ . A finite-size scaling collapse of the QSS lifetime data using the fitted exponents achieved a mean residual sum of squares 76% lower than a fixed-exponent ( $N^1$ ) collapse, confirming that the  $N$ -dependence of QSS lifetime is genuinely  $\alpha$ -dependent rather than a uniform finite-size artifact. These results provide quantitative numerical support for the applicability of  $q$ -statistics to the quasi-stationary states of long-range interacting systems specifically in the regime  $\alpha \leq 0.6$ , while showing that this non-extensive signature vanishes smoothly as the interaction range shortens toward the additive, short-range regime.

**Keywords:** Tsallis entropy; non-extensive statistical mechanics; long-range interactions;  $q$ -statistics; quasi-stationary states; mean-field models; power-law interactions; nonequilibrium relaxation

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## 1. INTRODUCTION

Boltzmann-Gibbs statistical mechanics rests on the assumption that a system's entropy and energy are additive over its subsystems, an assumption that holds for short-range interactions but breaks down for systems whose pairwise coupling decays slowly enough in space that the interaction energy of a subsystem with the rest of the system remains comparable to its self-energy even in the thermodynamic limit [1,2]. Gravitating systems, unscreened plasmas, and certain classes of cold-atom and dipolar systems are commonly cited physical examples of such long-range interacting systems, and their equilibration dynamics have long been observed to differ qualitatively from short-range systems, including markedly slower relaxation to final equilibrium [3,4].

Tsallis's non-extensive statistical mechanics generalizes the Boltzmann-Gibbs entropy to a one-parameter family indexed by  $q$ , reducing to the standard extensive entropy as  $q$  to 1, and has been proposed as a natural theoretical framework for systems where long-range coupling breaks additivity [5,6]. A substantial body of numerical work on the Hamiltonian mean-field (HMF) model, a paradigmatic fully-connected long-range spin/rotor system, has reported that such systems, when started from specific out-of-equilibrium initial conditions, pass through an extended quasi-stationary state before eventually relaxing to Boltzmann-Gibbs equilibrium, with the QSS lifetime reported to grow with system size in a manner suggestive of a genuinely non-extensive, rather than merely finite-size, effect [7,8,9].

Despite this substantial literature, the specific dependence of both QSS lifetime scaling and the quality of  $q$ -Gaussian fits to QSS momentum distributions on the interaction-range parameter  $\alpha$ , which continuously interpolates between fully long-range ( $\alpha = 0$ ) and short-range ( $\alpha$  large) coupling, has been less systematically mapped across a wide range of  $\alpha$  values within a single controlled numerical study, leaving open exactly how the non-extensive signature crosses over into ordinary extensive behavior as interaction range shortens [10,11,12].

This study addresses that gap by simulating the Hamiltonian mean-field model across six values of the interaction-range parameter  $\alpha$  spanning 0 to 1.5 and four to five system sizes at each  $\alpha$ , characterizing how QSS lifetime scaling with system size and the relative quality of  $q$ -Gaussian versus Maxwell-Boltzmann fits to the QSS momentum distribution both evolve as the interaction range shortens, in order to quantitatively delineate the regime of  $\alpha$  over which non-extensive statistical signatures are numerically robust.

## 2. MATERIALS AND METHODS

We performed  $N$ -body Hamiltonian dynamics simulations of the extended Hamiltonian mean-field model with power-law interaction decay  $r^{-\alpha}$ , tracking relaxation from an out-of-equilibrium water-bag initial condition to

final equilibrium across six values of  $\alpha$  and multiple system sizes at each  $\alpha$ , and fit both  $q$ -Gaussian and Maxwell-Boltzmann distributions to the momentum distribution during the quasi-stationary phase.

### 2.1 Model definition

The extended Hamiltonian mean-field model represents  $N$  particles as planar rotors with angle  $\theta_i$  and conjugate momentum  $p_i$ , coupled pairwise with interaction strength decaying as  $1/N_\alpha * r_{ij}^{-\alpha}$  for particle separation  $r_{ij}$  on a ring geometry, where  $N_\alpha$  is the Kac normalization factor ensuring extensive total energy at each  $\alpha$ , and  $\alpha$  in  $\{0, 0.2, 0.4, 0.6, 1.0, 1.5\}$  was tested, spanning fully long-range ( $\alpha = 0$ , equivalent to the standard mean-field HMF model) through short-range-like coupling ( $\alpha = 1.5$ ).

### 2.2 Initial condition and system sizes

All simulations were initialized in a water-bag out-of-equilibrium state, with particle angles uniformly distributed and momenta uniformly distributed within a fixed energy-density window known to produce extended quasi-stationary states in the  $\alpha = 0$  limit. For each of the six  $\alpha$  values, system sizes  $N = 500, 1,000, 2,000, 4,000$ , and  $8,000$  were simulated ( $N = 500$  and  $8,000$  omitted for the two largest  $\alpha$  values where computational cost of the long-range force sum became prohibitive), with 40 independent realizations per ( $\alpha, N$ ) combination to obtain averaged momentum distributions and lifetime estimates.

### 2.3 Numerical integration

Equations of motion were integrated using a fourth-order symplectic integrator with fixed timestep  $dt = 0.05$ , chosen to conserve total energy to better than 1 part in  $10^5$  over the full simulated trajectory length, with pairwise force sums computed directly (all-to-all) for  $N \leq 2,000$  and via a tree-based approximation with relative accuracy  $10^{-4}$  for larger  $N$  to keep computational cost tractable.

### 2.4 Quasi-stationary state identification and lifetime estimation

QSS lifetime was estimated for each realization as the time at which the single-particle kinetic energy distribution's excess kurtosis, tracked continuously, decayed to within 10% of its final equilibrium value, averaged across the 40 realizations per ( $\alpha, N$ ) combination; a power-law fit of  $\log(\text{lifetime})$  versus  $\log(N)$  was then performed at each  $\alpha$  to extract a scaling exponent and its 95% confidence interval from the regression standard error.

### 2.5 Distribution fitting

Momentum distributions sampled during the identified QSS phase (or, for  $\alpha$  values with no detectable QSS, during an equivalent early-time window) were fit by nonlinear least squares to both a  $q$ -Gaussian distribution,  $p(v)$  proportional to  $[1 - (1-q) \beta v^2]^{1/(1-q)}$ , and a Maxwell-Boltzmann ( $q = 1$ ) distribution, with fit quality

compared via reduced chi-squared statistics and the percentage reduction in reduced chi-squared achieved by the q-Gaussian fit relative to the Maxwell-Boltzmann fit.

## 2.6 Finite-size scaling collapse

For each alpha, QSS lifetime data across all tested  $N$  were rescaled using the fitted power-law exponent and compared, via residual sum of squares after collapse, against a fixed-exponent ( $N^1$ ) rescaling to test whether the alpha-dependent scaling exponents genuinely improved the data collapse relative to a uniform assumption.

## 3. RESULTS

For  $\alpha \leq 0.6$ , all tested system sizes exhibited a clearly identifiable extended quasi-stationary state preceding final relaxation to Boltzmann-Gibbs equilibrium, with QSS lifetime increasing systematically with system size at every alpha in this range. At  $\alpha = 0$  (fully long-range coupling), QSS lifetime scaled with system size as  $N^{1.68}$  (95% CI: 1.59–1.77), a clearly superlinear scaling consistent with the divergent-lifetime signature reported previously for the standard HMF model in the thermodynamic limit.

**Table 1. QSS lifetime scaling exponent and fitted q parameter by interaction-range parameter alpha.**

| alpha                  | QSS lifetime scaling exponent (95% CI) | Mean fitted q (SD) | QSS detected |
|------------------------|----------------------------------------|--------------------|--------------|
| 0 (fully long-range)   | 1.68 (1.59–1.77)                       | 1.42 (0.08)        | Yes          |
| 0.2                    | 1.31 (1.21–1.41)                       | 1.36 (0.07)        | Yes          |
| 0.4                    | 0.98 (0.87–1.09)                       | 1.24 (0.09)        | Yes          |
| 0.6                    | 0.71 (0.58–0.84)                       | 1.11 (0.06)        | Yes          |
| 1.0                    | -                                      | 1.02 (0.04)        | No           |
| 1.5 (short-range-like) | -                                      | 1.01 (0.03)        | No           |

Scaling exponent obtained from a power-law fit of QSS lifetime versus  $N$ ; not reported (-) for alpha values where no statistically distinguishable QSS plateau was detected.

The QSS lifetime scaling exponent decreased systematically as alpha increased across the long-range regime, from 1.68 at  $\alpha = 0$  to 1.31 at  $\alpha = 0.2$ , 0.98 at  $\alpha = 0.4$ , and 0.71 at  $\alpha = 0.6$ , indicating a smooth weakening of the anomalous lifetime scaling as interaction range shortened, summarized in Table 1. For  $\alpha \geq 1.0$ , no statistically distinguishable QSS plateau was detected at any tested  $N$ , with kurtosis relaxation times consistent with ordinary,  $N$ -independent short-range relaxation.



**Figure 1. Percentage reduction in fit quality (reduced chi-squared) achieved by the q-Gaussian distribution relative to the Maxwell-Boltzmann distribution, and mean fitted q parameter, as functions of the interaction-range parameter alpha, showing the crossover from non-extensive to extensive statistical behavior.**

Momentum distributions during the QSS phase were fit substantially better by a q-Gaussian than by a Maxwell-Boltzmann distribution at  $\alpha = 0$ , with mean fitted  $q = 1.42$  (SD 0.08) across the five system sizes tested and a 68% reduction in reduced chi-squared relative to the Maxwell-Boltzmann fit. This q-Gaussian fit advantage declined steadily with increasing alpha, falling to 34% at  $\alpha = 0.6$  and to a non-significant 4% ( $p = 0.31$ ) for  $\alpha \geq 1.0$ , tracing a clear crossover from non-extensive to ordinary extensive statistical behavior as summarized in Table 2 and Figure 1.

**Table 2. q-Gaussian versus Maxwell-Boltzmann fit quality to QSS momentum distributions by alpha.**

| alpha | Reduced chi-sq (q-Gaussian) | Reduced chi-sq (Maxwell-Boltzmann) | Reduction (%) | p-value |
|-------|-----------------------------|------------------------------------|---------------|---------|
| 0     | 1.12                        | 3.51                               | 68            | < 0.001 |
| 0.2   | 1.18                        | 2.94                               | 60            | < 0.001 |
| 0.4   | 1.24                        | 2.36                               | 47            | < 0.001 |
| 0.6   | 1.31                        | 1.98                               | 34            | 0.002   |
| 1.0   | 1.42                        | 1.48                               | 4             | 0.31    |
| 1.5   | 1.39                        | 1.43                               | 3             | 0.44    |

Across the  $\alpha \leq 0.6$  regime where QSS states were clearly present, the fitted  $q$  parameter itself declined systematically with alpha, from 1.42 at  $\alpha = 0$  to 1.36 at  $\alpha = 0.2$ , 1.24 at  $\alpha = 0.4$ , and 1.11 at  $\alpha = 0.6$ , approaching the Boltzmann-Gibbs limit  $q = 1$  continuously rather than discontinuously as interaction range shortened toward the short-range-like regime.

## 4. DISCUSSION

These results provide quantitative numerical support for the central claim motivating non-extensive statistical mechanics, that long-range interacting systems exhibit statistically distinct quasi-stationary states well described by q-statistics rather than Boltzmann-Gibbs statistics, extending earlier work largely confined to the  $\alpha = 0$  mean-field limit [5,6,7,13] by mapping how this signature weakens continuously as the interaction range parameter alpha increases toward the short-range regime, rather than persisting uniformly or vanishing abruptly at some threshold.

The continuous decline of both the fitted  $q$  parameter and the QSS lifetime scaling exponent toward their Boltzmann-Gibbs limits ( $q$  to 1, lifetime scaling exponent to 0) as  $\alpha$  increases is a secondary but mechanistically informative finding, since it suggests that non-extensivity in this class of models is not a binary property separating long-range from short-range systems but a smoothly graded one tied continuously to the degree of interaction-range-induced non-additivity, consistent with theoretical arguments that the Kac-normalized energy remains formally extensive at every  $\alpha$  while the fluctuation and relaxation properties vary continuously with  $\alpha$  [14,15,16].

More broadly, these findings have implications for identifying candidate physical systems where  $q$ -statistics descriptions are likely to be quantitatively, not merely qualitatively, useful: the sharp decline in  $q$ -Gaussian fit advantage beyond  $\alpha$  approximately 0.6 to 1.0 found here suggests that real long-range systems, such as self-gravitating systems or unscreened plasmas with effective interaction exponents in this intermediate range, may show only modest, and possibly experimentally difficult to distinguish, non-extensive signatures compared to the strongly non-extensive behavior seen at  $\alpha = 0$  [17,18,19,20].

Several limitations should be noted. First, all simulations were performed on the one-dimensional ring geometry of the extended HMF model; interaction-range crossover behavior could differ quantitatively, though plausibly not qualitatively, in higher-dimensional long-range systems. Second, only a single initial condition class (water-bag) and a single energy density were tested; QSS lifetime scaling and  $q$ -Gaussian fit quality are both known in the  $\alpha = 0$  literature to depend on initial-condition details, and this dependence was not mapped across  $\alpha$  here. Third, computational cost restricted the largest system size tested ( $N = 8,000$ ) to the four smallest  $\alpha$  values, so the finite-size scaling exponents reported for  $\alpha = 1.0$  and 1.5, where no QSS was detected at any available  $N$ , cannot rule out an extended QSS emerging only at system sizes beyond those accessible in this study [21,22,23,24,25,26].

## 5. CONCLUSION

$N$ -body simulations of the extended Hamiltonian mean-field model across six values of the interaction-range parameter  $\alpha$  and system sizes from 500 to 8,000 particles found that extended quasi-stationary states, well described by  $q$ -Gaussian rather than Maxwell-Boltzmann momentum distributions, persist clearly for  $\alpha \leq 0.6$ , with QSS lifetime scaling as  $N^{1.68}$  and mean fitted  $q = 1.42$  at the fully long-range limit  $\alpha = 0$ , both declining smoothly toward their Boltzmann-Gibbs limits as  $\alpha$  increases, before the non-extensive signature becomes statistically indistinguishable from ordinary extensive relaxation for  $\alpha \geq 1.0$ . These results give a quantitative,  $\alpha$ -resolved picture of where non-extensive statistical

mechanics offers genuine descriptive advantage over standard Boltzmann-Gibbs statistics in long-range interacting systems.

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## Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Author Contributions

R.S. conceived the study, designed the  $\alpha$  and system-size sweep, and led the writing of the manuscript. R.T. implemented the symplectic  $N$ -body integrator and tree-based force approximation and performed the simulation runs. S.U. performed the QSS lifetime and distribution-fitting analysis and prepared the figures and tables. All authors contributed to interpretation of the results and approved the final manuscript.

## Ethics Statement

This study involved no human or animal subjects and required no ethics committee approval; all data were generated by numerical simulation of a model long-range interacting particle system.

## Data Availability

The  $N$ -body simulation code, distribution-fitting scripts, and full  $\alpha$ /system-size sweep output data supporting the findings of this study are available from the corresponding author on reasonable request.

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