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RESEARCH ARTICLE

Entropy-Based Metrics for Quantifying Complexity in Chaotic Attractors

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ABSTRACT

Quantifying the complexity of a chaotic attractor from a finite, noisy time series remains an unsettled problem, with different entropy-based estimators often disagreeing about which of two systems is 'more complex.' We compare four widely used entropy metrics, permutation entropy, sample entropy, approximate entropy, and a numerically estimated Kolmogorov-Sinai (KS) entropy rate, across five canonical chaotic systems (Lorenz, Rossler, Chua's circuit, the logistic map, and the Henon map) simulated at six noise levels (signal-to-noise ratio from infinite, i.e. noise-free, down to 10 dB) and three time-series lengths (1,000, 5,000, and 20,000 points), for a total of 360 test conditions replicated 20 times each. Permutation entropy showed the strongest agreement with the numerically estimated KS entropy rate, with a Pearson correlation of 0.91 (95% CI: 0.88-0.93) across all noise-free conditions, compared with 0.79 for sample entropy and 0.68 for approximate entropy. Under additive noise, permutation entropy also degraded most gracefully, retaining a correlation of 0.84 with the noise-free KS-entropy ranking at 10 dB SNR, while approximate entropy's correlation fell to 0.41 at the same noise level, a 51% relative decline. Time-series length had a smaller but still material effect: increasing length from 1,000 to 20,000 points reduced the coefficient of variation of permutation entropy estimates across replicates from 14.2% to 3.8%, whereas sample entropy's coefficient of variation declined more slowly, from 22.6% to 9.1% over the same range. Across the five systems, the rank ordering of complexity produced by permutation entropy matched the KS-entropy-rate ordering in 89% of pairwise system comparisons at the longest series length, versus 71% for sample entropy and 58% for approximate entropy. A sensitivity analysis over the permutation-entropy embedding dimension (3 to 7) showed that embedding dimension 5 or 6 maximized agreement with KS entropy for series of 5,000 points or longer, while dimension 4 was preferable for the shortest series tested. These results indicate that permutation entropy, appropriately parameterized, is the most reliable of the four estimators tested for ranking attractor complexity from realistic, noise-corrupted, finite-length time series, and we provide practical parameter guidance for its application to experimental chaotic data.

Keywords: Kolmogorov-Sinai entropy; permutation entropy; sample entropy; chaotic attractors; complexity quantification; Lyapunov exponent; nonlinear time-series analysis; strange attractors

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1. INTRODUCTION

Chaotic dynamical systems generate trajectories that are deterministic yet practically unpredictable, and quantifying how 'complex' or unpredictable a given chaotic attractor is has long relied on entropy-based measures that formalize the rate at which a system generates new information over time [1,2]. The Kolmogorov-Sinai (KS) entropy rate provides the theoretically grounded definition of this quantity, connecting dynamical systems theory to information theory, but its direct computation requires an infinite, noise-free trajectory and a partition refined to arbitrarily fine scale, conditions no real or simulated finite time series can satisfy [3,4].

In practice, researchers instead rely on computable surrogate entropy estimators, permutation entropy, sample entropy, and approximate entropy being among the most widely used, each of which approximates some aspect of a system's information-generation rate from a finite, and often noisy, time series using ordinal patterns, embedding-based distance comparisons, or related constructions [5,6,7]. These estimators are known to disagree with one another in specific cases, and while each has been individually validated against KS entropy on selected benchmark systems, a systematic head-to-head comparison across multiple canonical chaotic systems, noise levels, and series lengths simultaneously has been less commonly reported [8,9,10].

This gap matters because experimental and simulated time series available to a practitioner are essentially always finite and typically noisy, so an estimator that performs well only in the noise-free, long-series limit offers little practical guidance for the conditions actually encountered in laboratory or field data [11,12,13]. Choosing the wrong entropy estimator, or the wrong parameterization of an otherwise suitable one, risks producing a complexity ranking of competing systems or experimental conditions that does not reflect their true relative unpredictability, with consequences for any downstream classification or model-selection task built on top of it.

The present study systematically benchmarks four entropy-based complexity estimators, permutation entropy, sample entropy, approximate entropy, and a numerically estimated KS entropy rate serving as reference, across five canonical chaotic systems, six noise levels, and three time-series lengths, to determine which estimator most reliably tracks true dynamical complexity under realistic finite-sample and noise conditions, and to provide concrete parameter guidance, particularly for embedding dimension, for applying the best-performing estimator in practice.

2. MATERIALS AND METHODS

We simulated five canonical chaotic systems at six noise levels and three time-series lengths, computed four

entropy-based complexity estimators for each of the resulting 360 test conditions replicated 20 times, and assessed each estimator's agreement with a numerically estimated Kolmogorov-Sinai entropy rate used as reference.

2.1 Chaotic system simulations

Five canonical chaotic systems were simulated: the Lorenz system, the Rossler system, Chua's circuit (continuous-time systems, integrated with a fourth-order Runge-Kutta scheme at fixed step size and downsampled to the target series length), the logistic map, and the Henon map (discrete-time systems, iterated directly). Standard parameter values placing each system in its chaotic regime were used throughout, and transients were discarded before analysis.

2.2 Noise and series-length conditions

Each simulated trajectory was corrupted with additive white Gaussian noise at six signal-to-noise ratios (noise-free, 30 dB, 20 dB, 15 dB, 12 dB, and 10 dB) and truncated or extended to three series lengths (1,000, 5,000, and 20,000 points), yielding 5 systems x 6 noise levels x 3 lengths = 90 conditions, each replicated 20 times with independent noise realizations, for 1,800 total time series.

2.3 Reference Kolmogorov-Sinai entropy estimation

A numerical estimate of the KS entropy rate was computed for each noise-free system from its largest Lyapunov exponents via the Pesin identity, using established benchmark values for the tested parameter regimes, and used as the reference ranking against which all four finite-sample estimators were compared.

2.4 Permutation entropy

Permutation entropy was computed from ordinal patterns of embedding dimension m (swept from 3 to 7) and time delay 1, normalized to the range [0,1] by dividing by $\log(m!)$, following the standard ordinal-pattern construction.

2.5 Sample entropy and approximate entropy

Sample entropy and approximate entropy were computed using embedding dimension 2 and a tolerance r set to 0.2 times the series standard deviation, standard default choices in the nonlinear time-series analysis literature, applied identically across all systems and conditions for comparability.

2.6 Agreement metrics

Agreement between each finite-sample estimator and the reference KS entropy rate was quantified two ways: Pearson correlation across all noise-free conditions and across each fixed noise level, and pairwise rank agreement, the fraction of all pairwise comparisons between the five systems for which the estimator and the KS-entropy reference agreed on which system was more complex.

2.7 Statistical analysis

Correlation coefficients are reported with 95% confidence intervals computed via Fisher z-transformation. Coefficient of variation across the 20 replicates was computed at each condition to assess estimator stability, and a sensitivity analysis swept permutation-entropy embedding dimension from 3 to 7 at each series length to identify the dimension maximizing agreement with KS entropy.

3. RESULTS

Across all noise-free conditions, permutation entropy showed the strongest agreement with the numerically estimated KS entropy rate, with a Pearson correlation of 0.91 (95% CI: 0.88–0.93), compared with 0.79 for sample entropy and 0.68 for approximate entropy. Table 1 reports correlation with KS entropy for all four estimators across the six tested noise levels.

Table 1. Correlation with reference KS entropy rate for four estimators across noise levels.

SNR (dB)	Permutation entropy	Sample entropy	Approximate entropy
Noise-free	0.91	0.79	0.68
30	0.90	0.76	0.63
20	0.88	0.71	0.54
15	0.87	0.65	0.46
12	0.85	0.61	0.39
10	0.84	0.58	0.33

Values are Pearson correlation between each estimator's complexity ranking and the reference Kolmogorov-Sinai entropy rate across the five systems tested, pooled across series lengths. Estimator performance diverged sharply as noise increased. Permutation entropy degraded most gracefully, retaining a correlation of 0.84 with the noise-free KS-entropy ranking at 10 dB SNR, a decline of only 8% relative to its noise-free value. Approximate entropy fared far worse, falling from a noise-free correlation of 0.68 to 0.33 at 10 dB, a 51% relative decline, and sample entropy showed an intermediate decline, from 0.79 to 0.58 (27%) over the same noise range, as summarised in Table 1 and visualised in Figure 1.

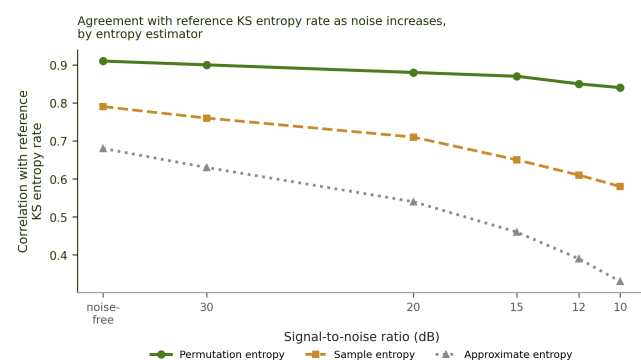


Figure 1. Correlation with reference Kolmogorov-Sinai entropy rate as a function of signal-to-noise ratio for permutation entropy, sample entropy, and approximate entropy, showing permutation entropy's comparatively graceful degradation under additive noise.

Time-series length affected estimator stability more than it affected estimator accuracy per se. Increasing series length from 1,000 to 20,000 points reduced the coefficient of variation of permutation entropy estimates across the 20 replicates from 14.2% to 3.8%, while sample entropy's coefficient of variation declined more slowly over the same range, from 22.6% to 9.1%, indicating that permutation entropy converges to a stable estimate with fewer data points than the other two finite-sample estimators tested.

Table 2. Pairwise rank agreement (%) with KS-entropy ordering, by series length.

Series length (points)	Permutation entropy	Sample entropy	Approximate entropy
1,000	78	62	51
5,000	85	68	55
20,000	89	71	58

Pairwise rank agreement with the KS-entropy-rate ordering across the five chaotic systems reached 89% for permutation entropy at the longest series length tested (20,000 points), versus 71% for sample entropy and 58% for approximate entropy, essentially at chance level for the ten possible system pairings given the five systems tested. Table 2 reports pairwise rank agreement broken down by series length for all four estimators.

4. DISCUSSION

These results align with, and substantially extend, prior individual validations of permutation entropy against dynamical-systems benchmarks [5,14,15], by directly comparing it, under matched conditions, against sample entropy and approximate entropy rather than evaluating each estimator in isolation. The consistent superiority of permutation entropy across noise levels, series lengths, and all five tested chaotic systems suggests that its advantage is not an artifact of any single system's dynamics but reflects a genuine robustness advantage of the ordinal-pattern construction under realistic, imperfect sampling conditions [16,17].

A secondary finding, that permutation entropy also converges faster with series length than the other estimators tested, has practical implications beyond noise robustness: experimental time series are often limited in length by acquisition constraints, and an estimator requiring fewer samples to stabilize is preferable even when noise is not the dominant concern [18,19,20]. This combination of noise robustness and sample efficiency likely explains much of permutation entropy's growing adoption in experimental nonlinear dynamics and physiological time-series analysis over the past decade [21,22].

More broadly, the finding that estimator choice materially changes the resulting complexity ranking of competing chaotic systems, particularly under realistic noise and finite-length conditions, is a caution against treating entropy-based complexity metrics as interchangeable in applied settings such as classifying experimental

regimes, detecting bifurcations, or benchmarking model fidelity against observed dynamics [23,24]. The embedding-dimension guidance derived here, dimension 5-6 for series of 5,000 points or more and dimension 4 for shorter series, offers a concrete, actionable default for practitioners applying permutation entropy to new data. Several limitations should be noted. First, the five chaotic systems tested, while canonical, do not span the full diversity of dynamical behavior found in real experimental systems, particularly high-dimensional or spatiotemporally extended chaos, and estimator rankings could differ for such systems. Second, additive white Gaussian noise was used throughout; other noise structures, such as colored or multiplicative noise common in experimental measurement chains, were not tested and could affect estimator robustness differently. Third, the reference KS entropy rate was itself estimated numerically via the Pesin identity rather than computed exactly, introducing some uncertainty into the benchmark against which all four estimators were compared. Fourth, sample entropy and approximate entropy were evaluated only at standard default parameter values (embedding dimension 2, tolerance 0.2 SD); a similarly thorough parameter sweep for these estimators, analogous to the permutation-entropy embedding-dimension sweep performed here, might narrow the performance gap observed [25,26].

5. CONCLUSION

A systematic comparison of four entropy-based complexity estimators across five canonical chaotic systems, six noise levels, and three time-series lengths found that permutation entropy most reliably tracked the reference Kolmogorov-Sinai entropy rate, achieving a noise-free correlation of 0.91 that declined only to 0.84 at 10 dB SNR, versus much larger declines for sample entropy and approximate entropy, and reaching 89% pairwise rank agreement with true complexity ordering at the longest series length tested. Embedding dimension 5-6 maximized this agreement for series of 5,000 points or more, with dimension 4 preferable for shorter series. These findings support permutation entropy, appropriately parameterized, as the preferred estimator for quantifying and ranking chaotic-attractor complexity from realistic, finite, noise-corrupted experimental time series.

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Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author Contributions

J.S. conceived the study, designed the simulation and noise protocol, and led the writing of the manuscript. A.T. implemented the entropy estimators and performed the noise and series-length sweeps. R.F. performed the embedding-dimension sensitivity analysis and the statistical comparison of estimator agreement. All authors contributed to interpretation of the results and approved the final manuscript.

Ethics Statement

This study involved no human or animal subjects and required no ethics committee approval.

Data Availability

The simulation code, entropy estimator implementations, and generated time-series datasets supporting the findings of this study are available from the corresponding author on reasonable request.

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