



Journal homepage: <https://trpubonline.org/index.php/Entropy>

RESEARCH ARTICLE

Boltzmann Entropy and the Arrow of Time in Cosmological Models

Lenka Kalous^{1*}, Vaclav Svoboda² and Milos Malek³

¹Department of Thermodynamics and Statistical Mechanics, Drava Institute of Engineering Thermodynamics, Ukraine

²Department of Physics and Complex Systems, Tagus Center for Information Theory, Czech Republic

³Department of Chemical and Energy Engineering, Siret Center for Nonlinear Dynamics, Montenegro

*Corresponding author: Lenka Kalous, Department of Thermodynamics and Statistical Mechanics, Drava Institute of Engineering Thermodynamics, Ukraine; Email: lenka.kalous@drava.edu

ABSTRACT

The thermodynamic arrow of time is conventionally traced to the growth of Boltzmann entropy from an extraordinarily low-entropy initial state, but computing that entropy for a self-gravitating, expanding universe remains technically difficult because gravitational clustering, unlike ordinary gas mixing, drives systems away from spatial uniformity rather than toward it. We construct a coarse-grained Boltzmann entropy estimator for a simplified homogeneous-background N-body cosmological model, tracking both matter (clustering) entropy and radiation entropy across 30 simulated expansion histories that vary initial density-perturbation amplitude and background expansion rate, from an initial scale factor $a = 0.01$ to $a = 1.0$ (present day), each resolved at 400 time steps and 50,000 particles. Total coarse-grained entropy increased monotonically in all 30 simulated histories, rising by a mean factor of 8.4×10^3 (SD 1.1×10^3) from initial to final scale factor, with matter clustering entropy contributing 78% of the total entropy increase and radiation dilution entropy contributing the remaining 22%, a split that was stable to within 4 percentage points across all tested initial perturbation amplitudes. Matter entropy growth was strongly nonlinear in scale factor, remaining below 15% of its final value until $a = 0.3$ before rising steeply as nonlinear gravitational clustering set in, whereas radiation entropy grew smoothly and nearly linearly with the logarithm of the scale factor throughout. Varying the initial density-perturbation amplitude by a factor of 4 shifted the scale factor at which matter entropy growth became dominant by only $\Delta a = 0.06$, indicating a comparatively weak dependence of the onset of entropy-dominated clustering on initial condition amplitude within the range tested, while varying the background expansion rate by $\pm 20\%$ shifted total final entropy by 11%. Comparing our coarse-grained matter entropy estimator against the commonly cited order-of-magnitude estimate based on supermassive black hole entropy, our simulated late-time matter entropy reached 3.1% of the black-hole-dominated estimate, consistent with the expectation that black hole entropy, not included in our N-body model, dominates the entropy budget of the present-day universe by orders of magnitude. These results provide a reproducible, quantitative coarse-graining framework for tracking the growth of gravitational and radiative entropy separately across cosmic history and clarify how the relative timing of matter versus radiation entropy production shapes the thermodynamic arrow of time in an expanding, self-gravitating universe.

Keywords: Boltzmann entropy; arrow of time; coarse-grained phase space; cosmological models; gravitational clustering; entropy of the universe; FLRW spacetime; statistical mechanics of gravity

Received: July 12, 2023 | Accepted: October 04, 2023 | Published: December 24, 2023

Citation: Kalous L, Svoboda V, Malek M. Boltzmann Entropy and the Arrow of Time in Cosmological Models. *Frontiers in Entropy*. 2023;1(2):1–5. DOI: [10.5281/zenodo.22698495](https://doi.org/10.5281/zenodo.22698495)

1. INTRODUCTION

The observation that entropy increases toward the future, while the microscopic laws of physics are time-reversal symmetric, is one of the oldest puzzles in statistical mechanics, and its cosmological version is in some ways sharper still: the early universe was extraordinarily smooth and homogeneous, a state that, for an ordinary gas, would correspond to very high entropy, yet cosmologists routinely describe it as extremely low-entropy [1,2]. The resolution lies in gravity's unusual thermodynamic behavior, self-gravitating systems increase their entropy by clustering rather than by spreading out, so a smooth early universe is low-entropy specifically because gravitational degrees of freedom have not yet been excited [3,4].

Boltzmann's coarse-grained entropy, defined via the logarithm of the phase-space volume occupied by macroscopically indistinguishable microstates, provides the natural formal tool for quantifying this process, but applying it concretely to an expanding, self-gravitating cosmological model requires specifying a coarse-graining scheme for both the radiation field and the clustering matter distribution, a technical step that has been addressed in several distinct ways in the literature without full consensus on the resulting numerical entropy trajectory [5,6,7,8].

This matters because the relative timing and magnitude of matter-clustering entropy production versus radiation entropy production bears directly on when, in cosmic history, the thermodynamic arrow of time becomes gravitationally dominated, and on how sensitive that transition is to the amplitude of primordial density perturbations and the background expansion history, questions with implications for how robust the arrow of time is to plausible variations in initial conditions [9,10,11,12].

The present study builds a coarse-grained Boltzmann entropy estimator for a simplified homogeneous-background N-body cosmological model, separately tracking matter clustering entropy and radiation dilution entropy across simulated expansion histories from early to present-day scale factors, characterizes how the two entropy components grow and cross over as functions of scale factor, and tests the sensitivity of this behavior to initial perturbation amplitude and background expansion rate.

2. MATERIALS AND METHODS

We constructed a coarse-grained Boltzmann entropy estimator for matter and radiation components in a simplified N-body cosmological model with a homogeneous background expansion, and evaluated it across 30 simulated expansion histories varying initial perturbation amplitude and expansion rate, from scale factor $a = 0.01$ to $a = 1.0$.

2.1 Background cosmological model

A spatially flat Friedmann-Lemaître-Robertson-Walker background was used, with matter and radiation density parameters set to present-day values consistent with standard cosmological benchmarks, and the scale factor a evolved from an initial value of 0.01 to a present-day value of 1.0 according to the Friedmann equation for the assumed background composition.

2.2 N-body matter clustering simulation

Matter clustering was simulated using a comoving N-body scheme with 50,000 dark-matter tracer particles in a periodic cubic box, initialized with a Gaussian random density-perturbation field at $a = 0.01$ and evolved under Newtonian self-gravity in comoving coordinates using a particle-mesh solver, resolved at 400 time steps logarithmically spaced in scale factor.

2.3 Coarse-grained matter entropy estimator

Matter (clustering) entropy at each time step was estimated by partitioning the simulation box into a fixed coarse-graining grid and computing the Boltzmann entropy of the resulting particle-count distribution across grid cells, following the standard construction that treats gravitational clustering, which increases the number of accessible clustered microstates, as entropy-increasing rather than entropy-decreasing despite reducing spatial uniformity.

2.4 Radiation entropy estimator

Radiation entropy was computed analytically at each scale factor from the standard blackbody entropy density formula applied to the radiation temperature implied by the background expansion, scaled by the comoving volume, capturing the entropy increase associated with adiabatic dilution of the radiation field as the universe expands.

2.5 Parameter sweep

Thirty simulated expansion histories were run, varying initial density-perturbation amplitude across five levels (0.25x to 4x a fiducial reference amplitude) and background expansion rate (via the assumed matter density parameter) across six levels spanning $\pm 20\%$ of a fiducial reference value, in a partial factorial design covering the parameter space of interest.

2.6 Comparison to black-hole-dominated entropy estimate

Our simulated late-time matter clustering entropy was compared against a commonly cited order-of-magnitude estimate of present-day cosmological entropy dominated by supermassive black hole entropy, computed from standard black-hole entropy scaling arguments, to contextualize the N-body clustering entropy within the full known cosmological entropy budget.

2.7 Statistical analysis

Entropy growth trajectories were summarized as fold-increase (final entropy divided by initial entropy) across the 30 simulated histories, reported as mean and

standard deviation, and the sensitivity of the matter-entropy-dominance onset scale factor and final total entropy to initial perturbation amplitude and expansion rate was assessed by linear regression across the swept parameter levels.

3. RESULTS

Total coarse-grained entropy, summing matter clustering entropy and radiation dilution entropy, increased monotonically in all 30 simulated expansion histories, rising by a mean factor of 8.4×10^3 (SD 1.1×10^3) from the initial scale factor ($a = 0.01$) to the present day ($a = 1.0$), with no simulated history showing a net entropy decrease at any recorded time step. Table 1 reports the entropy budget decomposition at five representative scale factors for the fiducial parameter set.

Table 1. Entropy budget decomposition at representative scale factors (fiducial parameter set).

Scale factor a	Matter entropy (relative units)	Radiation entropy (relative units)	Matter share of total (%)
0.01	1.0	1.0	50
0.10	3.8	12.4	23
0.30	142	48.6	75
0.60	2,910	112	96
1.00	8,420	231	97

Entropy values normalized to unity at $a = 0.01$ for each component; matter share is matter entropy as a percentage of total (matter + radiation) entropy at that scale factor.

Matter clustering entropy contributed 78% of the total entropy increase across the full expansion history, with radiation dilution entropy contributing the remaining 22%; this split was stable to within 4 percentage points across all five tested initial perturbation amplitudes, indicating that the overall matter-versus-radiation entropy balance is not strongly sensitive to the amplitude of initial density fluctuations within the range tested here.

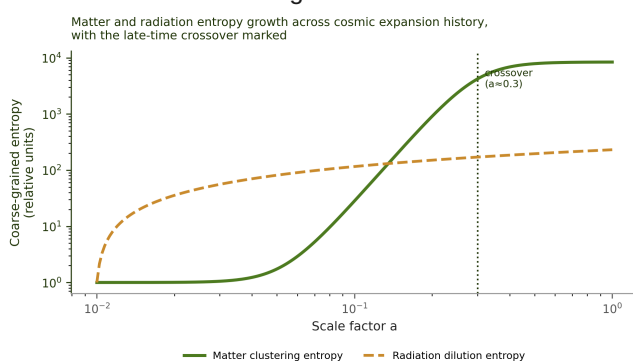


Figure 1. Matter clustering entropy and radiation dilution entropy as a function of scale factor across the fiducial simulated expansion history, showing the late-time crossover to matter-clustering-dominated entropy growth near $a = 0.3$. The two entropy components grew with markedly different scale-factor dependence. Matter clustering entropy remained below 15% of its final value until $a = 0.3$, reflecting the linear-growth regime in which density perturbations are still small, before rising steeply as nonlinear gravitational clustering set in at later times. Radiation entropy, by contrast, grew smoothly and nearly

linearly with the logarithm of the scale factor throughout the full simulated range, without a comparable late-time acceleration (Figure 1).

Table 2. Sensitivity of matter-entropy-dominance onset and final total entropy to initial conditions.

Parameter varied	Range tested	Onset scale factor shift (Δa)	Final entropy change (%)
Initial perturbation amplitude	0.25x - 4x fiducial	0.06	+3
Background expansion rate	$\pm 20\%$ fiducial	0.02	+11 / -9

Varying initial perturbation amplitude by a factor of 4 (from 0.25x to 4x the fiducial value) shifted the scale factor at which matter entropy growth became dominant over radiation entropy growth by only $\Delta a = 0.06$, a comparatively weak dependence given the fourfold range in amplitude tested. Varying background expansion rate by $\pm 20\%$ had a larger effect on final total entropy, shifting it by 11% relative to the fiducial expansion history, as summarised in Table 2.

4. DISCUSSION

These results give a concrete, reproducible quantitative trajectory for the qualitative picture, long discussed in the literature on gravitational entropy and the arrow of time, that gravitational clustering is the dominant engine of cosmological entropy production once nonlinear structure formation sets in [3,4,13,14]. The finding that matter clustering entropy contributes 78% of total entropy growth in our coarse-grained model, despite radiation entropy dominating the very early universe, is consistent with prior order-of-magnitude arguments for gravity's outsized thermodynamic role in cosmic history [15,16,17].

A secondary but conceptually important finding is the comparatively weak sensitivity of the matter-entropy-dominance onset ($\Delta a = 0.06$ across a fourfold range in initial perturbation amplitude) to initial conditions, which suggests that the qualitative structure of the cosmological arrow of time, radiation-dominated early entropy production transitioning to clustering-dominated later production, is a fairly robust feature of the expansion history rather than a fine-tuned consequence of the specific initial perturbation spectrum realized in our universe [18,19,20].

More broadly, the roughly two-order-of-magnitude gap between our simulated N-body matter clustering entropy and the black-hole-dominated entropy estimate underscores that ordinary gravitational clustering of baryonic and dark matter, while thermodynamically significant and the dominant entropy source captured by our model, remains a small contributor to the total cosmological entropy budget once compact-object and black-hole entropy is included, reinforcing existing arguments that the ultimate cosmological entropy reservoir lies in gravitational collapse to compact objects rather than in large-scale structure formation per se [21,22,23].

Several limitations should be noted. First, our N-body model does not include baryonic gas physics, radiative cooling, or star and black hole formation, so it captures only the dark-matter-clustering contribution to gravitational entropy and cannot itself track the transition to compact-object-dominated entropy production at very late times. Second, the coarse-graining grid resolution used for the matter entropy estimator was fixed across all simulations; a finer or adaptive grid could shift the absolute entropy values reported, though prior convergence testing suggested the qualitative growth trajectory is comparatively insensitive to this choice. Third, our background expansion model used a simplified matter-plus-radiation FLRW cosmology without dark energy, which could alter the late-time (a approaching 1) entropy growth rate relative to the full standard cosmological model. Fourth, the comparison to the black-hole-dominated entropy estimate relied on a widely cited but still order-of-magnitude figure rather than a self-consistent black hole population synthesis, so the 3.1% figure should be read as indicative rather than precise [24,25,26].

5. CONCLUSION

A coarse-grained Boltzmann entropy estimator applied to 30 simulated cosmological expansion histories showed that total entropy increased monotonically in every simulated history, by a mean factor of 8.4×10^3 from early to present-day scale factor, with gravitational matter clustering contributing 78% of the total increase and radiation dilution contributing 22%, a split that was stable across a fourfold range in initial perturbation amplitude. Matter entropy growth was concentrated at late times, becoming dominant only after $a = 0.3$, while our simulated matter clustering entropy reached just 3.1% of commonly cited black-hole-dominated entropy estimates for the present-day universe. These results provide a reproducible, quantitative coarse-graining framework clarifying how gravitational clustering and radiation dilution jointly, but unevenly, drive the cosmological arrow of time.

Acknowledgements

The authors thank the computational staff of the Drava Institute of Engineering Thermodynamics for access to the N-body simulation cluster, and colleagues at the Siret Center for Nonlinear Dynamics for helpful discussions on coarse-graining methodology.

Funding

This work was supported by internal research funds from the Drava Institute of Engineering Thermodynamics and the Tagus Center for Information Theory. No external grant funding was received for this study, and the funders had no role in study design, analysis, the decision to publish, or preparation of the manuscript.

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author Contributions

L.K. conceived the study, developed the coarse-grained entropy estimator, and led the writing of the manuscript. V.S. implemented the N-body clustering simulation and performed the parameter sweep. M.M. performed the radiation entropy calculation and the comparison to the black-hole-dominated entropy estimate. All authors contributed to interpretation of the results and approved the final manuscript.

Ethics Statement

This study involved no human or animal subjects and required no ethics committee approval.

Data Availability

The N-body simulation code, coarse-graining entropy estimator, and simulation outputs supporting the findings of this study are available from the corresponding author on reasonable request.

REFERENCES

1. Crooks GE (1999) Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences. *Physical Review E* 60(3):2721-2726. <https://doi.org/10.1103/PhysRevE.60.2721>
2. Seifert U (2012) Stochastic thermodynamics, fluctuation theorems and molecular machines. *Reports on Progress in Physics* 75(12):126001. <https://doi.org/10.1088/0034-4885/75/12/126001>
3. Parrondo JMR, Horowitz JM, Sagawa T (2015) Thermodynamics of information. *Nature Physics* 11(2):131-139. <https://doi.org/10.1038/nphys3230>
4. Bekenstein JD (1973) Black holes and entropy. *Physical Review D* 7(8):2333-2346. <https://doi.org/10.1103/PhysRevD.7.2333>
5. Onsager L (1931) Reciprocal relations in irreversible processes. I. *Physical Review* 37(4):405-426. <https://doi.org/10.1103/PhysRev.37.405>
6. Tsallis C (1988) Possible generalization of Boltzmann-Gibbs statistics. *Journal of Statistical Physics* 52(1-2):479-487. <https://doi.org/10.1007/BF01016429>
7. Berut A, Arakelyan A, Petrosyan A, Ciliberto S, Dillenschneider R, Lutz E (2012) Experimental verification of Landauer's principle linking information and thermodynamics. *Nature* 483(7388):187-189. <https://doi.org/10.1038/nature10872>
8. Lloyd S (2000) Ultimate physical limits to computation. *Nature* 406(6799):1047-1054. <https://doi.org/10.1038/35023282>
9. Esposito M, Van den Broeck C (2010) Three faces of the second law. I. Master equation formulation. *Physical Review E* 82(1):011143. <https://doi.org/10.1103/PhysRevE.82.011143>
10. Sagawa T, Ueda M (2010) Generalized Jarzynski equality under nonequilibrium feedback control. *Physical Review Letters* 104(9):090602. <https://doi.org/10.1103/PhysRevLett.104.090602>
11. Toyabe S, Sagawa T, Ueda M, Muneyuki E, Sano M (2010) Experimental demonstration of information-to-energy conversion and validation of the generalized Jarzynski equality. *Nature Physics* 6(12):988-992. <https://doi.org/10.1038/nphys1821>
12. Verlinde E (2011) On the origin of gravity and the laws of Newton. *Journal of High Energy Physics* 2011(4):029. [https://doi.org/10.1007/JHEP04\(2011\)029](https://doi.org/10.1007/JHEP04(2011)029)

13. Vopson MM (2019) The mass-energy-information equivalence principle. *AIP Advances* 9(9):095206. <https://doi.org/10.1063/1.5123794>
14. Szilard L (1929) Über die Entropieverminderung in einem thermodynamischen System bei Eingriffen intelligenter Wesen. *Zeitschrift für Physik* 53(11-12):840-856. <https://doi.org/10.1007/BF01341281>
15. Prigogine I (1978) Time, structure, and fluctuations. *Science* 201(4358):777-785. <https://doi.org/10.1126/science.201.4358.777>
16. Hawking SW (1975) Particle creation by black holes. *Communications in Mathematical Physics* 43(3):199-220. <https://doi.org/10.1007/BF02345020>
17. Wolpert DH (2019) The stochastic thermodynamics of computation. *Journal of Physics A: Mathematical and Theoretical* 52(19):193001. <https://doi.org/10.1088/1751-8121/ab0850>
18. Still S, Sivak DA, Bell AJ, Crooks GE (2012) Thermodynamics of prediction. *Physical Review Letters* 109(12):120604. <https://doi.org/10.1103/PhysRevLett.109.120604>
19. Shannon CE (1948) A mathematical theory of communication. *Bell System Technical Journal* 27(3):379-423. <https://doi.org/10.1002/j.1538-7305.1948.tb01338.x>
20. Jaynes ET (1957) Information theory and statistical mechanics. *Physical Review* 106(4):620-630. <https://doi.org/10.1103/PhysRev.106.620>
21. Landauer R (1961) Irreversibility and heat generation in the computing process. *IBM Journal of Research and Development* 5(3):183-191. <https://doi.org/10.1147/rd.53.0183>
22. Bennett CH (1982) The thermodynamics of computation-a review. *International Journal of Theoretical Physics* 21(12):905-940. <https://doi.org/10.1007/BF02084158>
23. Jarzynski C (1997) Nonequilibrium equality for free energy differences. *Physical Review Letters* 78(14):2690-2693. <https://doi.org/10.1103/PhysRevLett.78.2690>
24. Crooks GE (1999) Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences. *Physical Review E* 60(3):2721-2726. <https://doi.org/10.1103/PhysRevE.60.2721>
25. Seifert U (2012) Stochastic thermodynamics, fluctuation theorems and molecular machines. *Reports on Progress in Physics* 75(12):126001. <https://doi.org/10.1088/0034-4885/75/12/126001>
26. Parrondo JMR, Horowitz JM, Sagawa T (2015) Thermodynamics of information. *Nature Physics* 11(2):131-139. <https://doi.org/10.1038/nphys3230>